Stochastic processes – written exam
27.1.2020.

1. A rich man is playing a series of games. The stake in $n-$th game is 1 or 2 dollars, depending on the outcome in the previous game. If he lost $(n-1)-$st game, the stake in $n-$th is 1 and the probability that he wins the $n-$th game is $\frac{1}{2}$. In that case he earns additional 1 dollar. Otherwise, he losses 1 dollar he invested. If he won $(n-1)-$st game the stake in $n-$th is 2 and the probability that he wins the $n-$th game is $\frac{1}{3}$. In that case he earns additional 2 dollars. Otherwise, he losses 2 dollars he invested. Denote by X_n the random variable representing his gain in $n-$th game,

$$X_n : \begin{pmatrix} -2 & -1 & 1 & 2 \\ p_1 & p_2 & p_3 & p_4 \end{pmatrix}, \quad n \in \mathbb{N}_0.$$

X_0 denotes the initial state. Let $Y_n = \operatorname{sgn} X_n$, $n \in \mathbb{N}_0$.

(a) Find $E(X_n|Y_{n-1})$.

(b) Find the one step transition probability matrix of the Markov chain $\{Y_n\}$. What is the proportion of games the rich man losses?

(c) Is Markov chain $\{Y_n\}$ stationary? Explain.

(d) Calculate the probabilities p_i, $i = 1, 2, 3, 4$ and $E(X_n)$.

2. The occurrence times of the death of the policyholders of a certain life-insurance company are treated as an arrival times of insurance claims and the number of deaths is treated as Poisson process X_t with rate 2 per day. Denote by Z_n the amount of claims that the $n-$th policyholder that passed away carries. Suppose that random variables Z_n, $n = 1, 2, \dots$ are independent and uniformly distributed between 1000 and 5000 euros. Find the expected amount and variance of claims that the insurance company will have to pay during 5 days period.

3. Consider the stochastic process $X_t = 2W_t + \sqrt{t}$, where W_t denotes standard Brownian motion.

(a) Find autocovarianse function of the stochastic process X_t.

(b) Check if the following is true

$$E(X_t^2|\mathcal{W}_s) = X_s^2, \quad s \in A = \{x > 0 : F_{X_t}(x) < 0.5\}.$$

(c) Is the process X_t^2 martingale with respect to history of Brownian motion $\mathcal{W}_t$?