Markov chains

Example 1. [A Communications System] Consider a communications system which transmits the digits 0 and 1. Each digit transmitted must pass through several stages, at each of which there is a probability p that the digit entered will be unchanged when it leaves. Letting X_n denote the digit entering the $n-$th stage, then $\{X_n, n = 0, 1, \ldots\}$ is a two-state Markov chain having a transition probability matrix

$$P = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}.$$

Example 2. On any given day Gary is either cheerful (C), so-so (S), or glum (G). If he is cheerful today, then he will be C, S, or G tomorrow with respective probabilities 0.5, 0.4, 0.1. If he is feeling so-so today, then he will be C, S, or G tomorrow with probabilities 0.3, 0.4, 0.3. If he is glum today, then he will be C, S, or G tomorrow with probabilities 0.2, 0.3, 0.5. Letting X_n denote Gary's mood on the $n-$th day, then $\{X_n, n \geq 0\}$ is a three-state Markov chain (state $0 = C$, state $1 = S$, state $2 = G$) with transition probability matrix

$$P = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}.$$

Problem 1. In a two-state discrete-time Markov chain, state changes can occur each second. Once the system is OFF, the system stays off for another second with probability 0.2. Once the system is ON, it stays on with probability 0.1. Find the state transition matrix P.

Problem 2. Each second, a laptop computer's wireless LAN card reports the state of the radio channel to an access point. The channel may be (0) poor, (1) fair, (2) good, or (3) excellent. In the poor state, the next state is equally likely to be poor or fair. In states 1, 2, and 3, there is a probability 0.9 that the next system state will be unchanged from the previous state and a probability 0.04 that the next system state will be poor. In states 1 and 2, there is a probability 0.06 that the next state is one step up in quality. When the channel is excellent, the next state is either good with probability 0.04 or fair with probability 0.02. Find the state transition matrix P.

Problem 3. Consider the Markov chain consisting of the three states and having transition probability matrix

$$P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/3 & 1/6 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}.$$

1. Is the Markov chain homogeneous?
2. What is p_{13} and p_{23}?
3. Find $p_{13}(2)$
4. Find P_2

Problem 4. The one-step transition probability matrix of the Markov chain is given by

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}.$$

Use the Chapman-Kolmogorov equations to determine three-step transition probability that starting in the state 1 will end up in the state 2.

Problem 5. Consider Markov chain with the one-step transition probability matrix

$$P = \begin{bmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}.$$

(a) Determine in how many steps one can make a transition from a state 2 to state 3.

(b) Determine in how many steps one can make a transition from a state 2 to state 4.

Problem 6. Prove that a Markov chain with one-step transition matrix $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is not ergodic.

Eigenvalues and eigenvectors of a matrix

For a given square matrix A a **characteristic polynomial** is $p_A(\lambda) = \det(\lambda I - A)$, where I is identity matrix. The spectrum of a matrix A, $\sigma(A) = \{\lambda \in \mathbb{C} : p_A(\lambda) = 0\}$ is the set of its eigenvalues i.e. $\lambda \in \mathbb{C}$ is an eigenvalue of a matrix A if and only if $\lambda \in \sigma(A)$. Vector $x \in \ker(\lambda I - A)$ is an eigenvector corresponding to the eigenvalue λ. The **right eigenvector** is a column vector satisfying $\lambda x = Ax$ and the **left eigenvector** is a row vector satisfying $\lambda x = xA$.
Cayley Hamilton theorem: $p_A(A) = 0$.

The eigendecomposition (or spectral decomposition) of a diagonalizable matrix A is a decomposition of a diagonalizable matrix into a specific canonical form whereby the matrix is represented in terms of its eigenvalues and eigenvectors. Suppose that the eigenvalues of a $n \times n$ matrix A are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$. The right eigenvector corresponding to λ_i is denoted by r_i and the left eigenvector corresponding to λ_i is denoted by l_i. So,

$$\lambda_i r_i = A r_i, \quad \lambda_i l_i = l_i A.$$

It holds

$$\det(A) = \lambda_1 \lambda_2 \cdot \ldots \cdot \lambda_n, \quad \operatorname{tr}(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n.$$

Then

$$A = S^{-1} D S = T D T^{-1},$$

where $D = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, the $i-$th row of a matrix S is l_i, while the $i-$th column of a matrix T is r_i. Then

$$A^n = S^{-1} D^n S = T D^n T^{-1}.$$

Problem 7. The state of a discrete-time Markov chain with transition matrix P can change once each second; X_n denotes the system state after n seconds. An observer examines the system state every m seconds, producing the observation sequence $\hat{X}_0, \hat{X}_1, \ldots$ where $\hat{X}_n = X_{mn}$. Is $\hat{X}_0, \hat{X}_1, \ldots$ a Markov chain? If so, find the state transition matrix $\hat{P}$.

Problem 8. The two-state Markov chain can be used to model a wide variety of systems that alternate between ON and OFF states. After each unit of time in the OFF state, the system turns ON with probability p. After each unit of time in the ON state, the system turns OFF with probability q. Using 0 and 1 to denote the OFF and ON states, find a two state transition matrix. What is the probability the system is OFF at time $n = 33$?

Problem 9. A person's position is marked by an integer on the real line. Each unit of time the person stays on the same place or randomly moves one step, either to the right or to the left, all with probabilities 1/3. It is not important how the person came on that place.

(a) Find a two-step transition probabilities.

(*b*) If X_n denotes state of the person in the $n-$th unit of time, find $E(X_3|X_1 = 1)$.

Problem 10. A packet voice communications system transmits digitized speech only during "talk-spurts" when the speaker is talking. In every $10-$ms interval (referred to as a timeslot) the system decides whether the speaker is talking or silent. When the speaker is talking, a speech packet is generated; otherwise no packet is generated. If the speaker is silent in a slot, then the speaker is talking in the next slot with probability $p = 1/140$. If the speaker is talking in a slot, the speaker is silent in the next slot with probability $q = 1/100$. If states 0 and 1 represent silent and talking, find the state transition matrix P for this packet voice system. What is the limiting state probability vector $[p_0^* \;\; p_1^*]$?

Problem 11. Find the limiting state probabilities for the Markov chain with one-step transition probability matrix $P = \begin{bmatrix} 1/4 & 3/4 \\ 1/3 & 2/3 \end{bmatrix}$.

Problem 12. Each of two switches is either on or off during a day. On day n, each switch will independently be off with probability

$$\frac{1 + \text{ number of on switches during day } (n-1)}{4}.$$

For instance, if both switches are on during day $n-1$, then each will independently be off during day n with probability 3/4. What fraction of days are both switches on? What fraction are both off?

Problem 13. A DNA nucleotide has any of 4 values. A standard model for a mutational change of the nucleotide at a specific location is a Markov chain model that supposes that in going from period to period the nucleotide does not change with probability $1-3\alpha$, and if it does change then it is equally likely to change to any of the other 3 values, for some $0 < \alpha < \frac{1}{3}$.

(*a*) Show that $p_{11}(n) = \dfrac{1}{4} + \dfrac{3}{4}(1-4\alpha)^n$.

(*b*) What is the long run proportion of time the chain is in each state?

Problem 14. Consider a Markov chain with two state transition probability matrix

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}.$$

(*a*) The initial state vector is given by $p(0) = [2/3 \quad 1/3]$. Is this Markov chain stationary?

(*b*) Check if this Markov chain is ergodic and if it is find the limiting state probability vector.

Problem 15. A professor continually gives exams to her students. She can give three possible types of exams, and her class is graded as either having done well or badly. Let p_i denote the probability that the class does well on a type i exam, and suppose that $p_1 = 0.3$, $p_2 = 0.6$, and $p_3 = 0.9$. If the class does well on an exam, then the next exam is equally likely to be any of the three types. If the class does badly, then the next exam is always type 1. What proportion of exams are type i, $i = 1, 2, 3$?

Problem 16. A transition probability matrix for a Markov chain is given by

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(*a*) Prove that this chain is not ergodic.

(b) Find the limiting state probabilities $\lim_{n\to\infty} p_{ij}(n)$.

Problem 17. Customers in a certain city are continually switching the brand of soap they buy. If a customer is now using brand A, the probability he will use brand A next week is 0.5, that he switches to brand B is 0.2 and that he switches to brand C is 0.3. If he now uses brand B, the probability he uses B next week is 0.6 and the probability that he switches to C is 0.4 . If he now uses brand C, the probability he uses C next week is 0.4, that he switches to A is 0.2 and to B is 0.4. Assume the process is a Markov Chain.

(a) Find the probability a customer now using brand A will be using brand B in two weeks.

(b) If the percentage of customers now using brand A is 30%, the percentage using brand B is 20% and the percentage using brand C is 50%, find the percentage of customers using brand C in three weeks.

Problem 18. The diffusion of electrons and holes across a potential barrier in an electronic devise is modeled as follows. There are m black balls (electrons) in urn A and m white balls (holes) in urn B. We perform independent trials, in each of which a ball is selected at random from each urn and the selected ball from urn A is placed in urn B, while that from urn B is placed in A. Consider the Markov chain representing the number of black balls in urn A immediately after the $n-$th trial.

(a) Describe the one-step transition probabilities of the process.

(b) Suppose $m = 2$. Compute the long-run fraction of time when urn A does not contain a black ball.

Problem 19. Consider an urn initially containing r red and b black balls. Repeated drawings are made from this urn as follows: after each drawing one returns a ball and adds a balls of the same color. Here r, b and a are positive integers. Let $\{Y_n\}$ be a sequence of random variables such that $Y_n = 1$ if the $n-$th ball drawn is red and $Y_n = 0$ if the $n-$th ball drawn is black. Let r_n and b_n be the number of red and black balls, respectively in the urn after the $n-$th draw has been completed. Define Z_n as the number of red balls at the completion of the $n-$th draw. Express Z_n as a function of a random variables $Y_1, \ldots, Y_n$. Prove that $\{Z_n\}$ is Markov chain and find the one step transition probabilities. Is $\{Z_n\}$ homogeneous or non-homogeneous?