Stochastic processes – written exam
3.9.2018.

1. Let $X : \mathcal{N}(7,4)$ be the random variable that represents a lifetime of expensive devise. If the device breaks during the first two years the amount of x dinars is paid to insurer, while if it breaks during the third or the fourth year the amount of $\frac{1}{2}x$ dinars is paid to the insurer. The insurance is paid only if the lifetime of the device is less then 4 years. Find the value for x such that the expected payment per device is 5000 dinars.

2. (a) Nikola starts each month with 80 000 dinars on the bank account. On the first day of the month he pays the bills (amount is 10 000 dinars), on the second and the fifteenth day of the month he goes to the supermarket. Amount of money that he spends on those days is a random variable

$$X : \begin{pmatrix} 5000 & 7000 & 8000 & 10000 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}.$$

Amount of money spent on any other day is uniformly distributed in the interval $(0, 2000)$. Denote by X_n the total amount of money available on the $n-$th day of the month. Suppose that a month has 30 days. Find the expected amount of money available on the nth day of the month if an amount available on $(n-1)-$st day is known and find the expected amount of money spent on arbitrary chosen day.

(b) Suppose that on average Nikola spends more than 1000 dinars each two days. What is the probability that Nikola spends more than 1000 dinars less than 2 times in 3 days?

3. Peter takes the course Stochastic Processes this semester on Tuesday, Thursday and Friday. The classes start at 10 am. Peter is used to work until late in the night and consequently, he sometimes misses the class. His attendance behaviour is such that he attends class depending only on whether or not he went to the latest class. If he attended class one day, then he will go to class next time it meets with probability $1/2$. If he did not go to one class, then he will go to the next class with probability $3/4$.

(a) Describe the Markov chain that models Peter's attendance. What is the probability that he will attend class on Thursday if he went to class on Friday?

(b) Find the probability that Peter attends a class?

(c) Suppose the course has 30 classes altogether. Give an estimate of the number of classes attended by Peter and explain it.

4. Consider the sequence $X_1, X_2, \ldots$ of independent integrable identically distributed random variables such that

$$\phi(\lambda) = E[e^{\lambda X_1}] < +\infty, \qquad \text{for some } \lambda \neq 0.$$

Prove that

$$M_n := \phi^{-n}(\lambda) e^{\lambda S_n}, \qquad \text{where } S_n := \sum_{i=1}^{n} X_n$$

is martingale with respect to filtration $\mathcal{F}_n = \mathcal{F}(X_1, \ldots, X_n)$.