Stochastic processes – written exam
11.9.2019.

1. Let $X_t = -2 + (t+1)Y$, $t \in \mathbb{R}$, where

$$Y : \begin{pmatrix} 1 & 2 & 3 \\ 1/3 & 1/4 & 5/12. \end{pmatrix}$$

Find the first order distribution function $F_t(x)$ of stochastic process X_t. For which values of x the following holds $F_0(x) = 7/12$?

2. A class has n students and the quiz score of student i is x_i. The students are divided into k disjoint subsets $A_1, \dots, A_k$ and accordingly assigned to different sections. Let n_s denote the number of students in section s. Then the average (expected) score in section s is

$$m_s = \frac{1}{n_s} \sum_{i \in A_s} x_i.$$

Find the expected quiz score of an arbitrary chosen student and prove that obtained value does not depend on m_s and n_s, $s = 1, \dots, k$.

3. The one step transition probability matrix of Markov chain is given by

$$P = \begin{bmatrix} 0 & 1 & 0 \\ b & 0 & c \\ 0 & 1 & 0 \end{bmatrix},$$

where $b, c > 0$. Denote by $E = \{0, 1, 2\}$ the corresponding set of states.

(a) Is Markov chain ergodic? What can be concluded about existence of limiting state probabilities?

(b) Find the initial state vector such that Markov chain is stationary.

(c) List all transient states from the set E.

4. (a) Suppose that a company claims two licenses (A and B) for cars which production follows Poisson process with rate 1 per day. If it is known that one in 10 cars is sent back to the production due to final test failure, find the probability that more then one car is sent back to the production in 10 days.

(b) If two category of cars are produced independently but both productions follow Poisson process with rate 0.5 per day, find the probability that no more then one car (no matter the category) is produced during the period of 10 days.

5. Consider a sequence of independent identically distributed and bounded random variable $X_1, X_2, \dots$ (that is, $|X_k| \le a < \infty$). Define $S_n = \sum_{k=1}^{n} X_k$. Find the sufficient condition for $S_n^2 - E(S_n^2)$ to be martingale with respect to filtration $\mathcal{F}_n$, where $\mathcal{F}_n = \mathcal{F}(X_1, X_2, \dots, X_n)$.