Stochastic processes – written exam

17.9.2018.

1. Consider the following joint probability density function

$$\varphi_{X,Y}(x,y) = \begin{cases} Cye^{-x}, & x,y \in (0,1), \\ 0, & x,y \notin (0,1). \end{cases}$$

(a) Determine the value of C and then determine the marginal probability density functions for X and Y.

(b) Find the autocovariance function of stochastic process $X_t = X + tY$.

2. A game is played as follows. The integers $N \geq 2$ and $s \leq N-1$ are randomly chosen. There are two types of balls in the box: white and black. Initially, the box is filled with N balls. The number of white balls is s. Trials are performed as follows. In each trial one ball is selected. If that ball is white, one returns the selected ball in the box. If the ball is black, one returns the selected ball and adds white ball in the box. Consider the Markov chain representing the number of white balls in the box after the $n-$th trial.

(a) Find the one and two step transition probabilities of the Markov chain.

(b) The game ends when the probability that a number of white balls in the box after d trials increases by d is not greater then $1/30$. What is the minimum number of trials necessary for that to happen if $N=2$, $s=1$?

3. Suppose that arrivals to the barber shop occur according to Poisson process with rate 2 per hour. Each customer wants to get his hair cut with probability $2/3$, or to get shaved with probability $1/3$.

(a) Determine the probability that the 4 customers arrived to barber shop to get a haircut in the first 4 hours if it is known that 2 customers arrived to barber shop to get a haircut in the first 3 hours.

(b) Find the expected time between two consecutive arrivals of customers with the request for haircut.

4. (a) Find the function $f = f(s,t,x)$ such that the following holds

$$P\{W_t < 5\sqrt{t}\}E[W_t^2 W_s^2 | \mathcal{W}_s] + W_s^2 E[W_{2s}|\mathcal{W}_s] = W_s^2 f(s,t,W_s), \quad 0 < s \leq t,$$

if W_t is standard Brownian motion and $\mathcal{W}_t$ is history of Brownian motion until time t.

(b) Determine $E[f(s,t,W_s)]$.