Stochastic processes – written exam
17.6.2019.

1. Find the constant $c \in \mathbb{R}$ such that the function

$$\varphi(x) = c|x|e^{-x^2},\ x \in \mathbb{R}$$

is probability density function of some random variable X and determine $D(X)$.

2. The game of chance is played as follows. The machine randomly and one by one displays digits 0 and 1 on the screen, with probabilities $1-p$ and p, respectively. The game ends when the first zero appears. If your stake in one game is $a > 0$, the winning in that game equals the sum of all digits 0 and 1 shown on the screen. What is the expected amount of money you will win in one game? Find the value of p such that the game is fair (winnings are equal to losses).

3. Consider a mechanical device in which "shocks" occur according to a Poisson process with rate 0.1 per hour. The device fails when a total of K shocks occurs. Find the expected lifetime T of the device. Find the probabilities that K and $K-1$ shocks have occurred after $E(T)$ hours and compare those two values.

4. Consider an urn initially containing r red and b black balls. Repeated drawings are made from this urn as follows: after each drawing one returns a ball and adds a balls of the same color. Here r, b and a are positive integers. Let $\{Y_n\}$ be a sequence of random variables such that $Y_n = 1$ if the $n-$th ball drawn is red and $Y_n = 0$ if the $n-$th ball drawn is black. Let r_n and b_n be the number of red and black balls, respectively in the urn after the $n-$th draw has been completed.

(a) Define Z_n as the number of red balls at the completion of the $n-$th draw. Express Z_n as a function of a random variables $Y_1, \ldots, Y_n$. Prove that $\{Z_n\}$ is Markov chain and find the one step transition probabilities. Is $\{Z_n\}$ homogeneous or non-homogeneous?

(b) Define X_n as the proportion of red balls in the urn at the completion of the $n-$th draw, that is $X_n = \frac{r_n}{r_n+b_n}$. Prove that $\{X_n\}$ is a martingale with respect to the filtration $\{\mathcal{F}_n\}$, where $\mathcal{F}_n = \mathcal{F}(Y_1, \ldots, Y_n)$.