WRITTEN EXAM – STOCHASTIC PROCESSES, 28.8.2017.

1. At noon on a weekday, we begin recording new call attempts at a telephone switch. Let X denote the arrival time of the first call, as measured by the number of seconds after noon. Let Y denote the arrival time of the second call. In the most common model used in the telephone industry, X and Y are continuous random variables with joint probability density function

$$\varphi_{X,Y}(x,y) = \begin{cases} \lambda^2 e^{-\lambda y}, & 0 \le x < y, \\ 0, & otherwise, \end{cases}$$

where $\lambda > 0$ calls/second is the average arrival rate of telephone calls. Find the marginal probability density function of Y and the conditional probability density function $\varphi_{Y|\{X=x\}}(y)$.

2. Each day a system is gathering a certain amount of data and sums the money earned that day. Then a program classifies the previous day into a category. Each day can be classified as a Type i, $i \in [1,k]$. It is known that probability the day is classified as Type i is p_i and that $\sum_{i=1}^{k} p_i^2 = 0.5$. Also, the expected amount of money earned during the Type i day is $\ln\left(\frac{a}{e^{p_i}}\right)$ millions, $a > 1$. What is the expected amount of money earned on arbitrary chosen day?

3. The social status of the $n-$th generation of some family is given by the Markov chain $X_n, n \in \mathbb{N}_0$. Some family can belong to one of three social classes: 1 - lower, 2 - middle and 3 - upper. The transition probability matrix which describes the change in classes is given by

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}.$$

(a) Find a probability that a person whose parents are in the middle class ends up in the upper class and that the children of that person are in the lower class.

(b) Determine a probability that the children of the person whose parents are in the middle class end up in the lower class?

4. The number of trains from Novi Sad to Belgrade which are late more than one hour is described by a Poisson process with a rate one per month, while number of trains from Novi Sad to Subotica which are late more than one hour is an independent Poisson process with a rate 2 per month.

(a) What is the probability that at least 3 trains from Novi Sad to Belgrade are late more then one hour during one month?

(b) What is the probability that 3 trains from Novi Sad to Belgrade and 2 trains from Novi Sad to Subotica are late more that 1 hour during 1 month?

5. $X_1, X_2, \ldots$ is a sequence of independent identically distributed random variables with zero expectation and finite variance. Suppose that X_i cannot take absolute values greater than 1. Define

$$S_n := X_1 + \ldots + X_n, \quad n \in \mathbb{N}.$$

Find a constant a such that $S_n^2 - aE(S_n^2)$ is a martingale with respect to a filtration $\mathcal{F}_n$ which is a $\sigma-$algebra generated by $X_1, \ldots, X_n$.