Markov chains

1. In a two-state discrete-time Markov chain, state changes can occur each second. Once the system is OFF, the system stays off for another second with probability 0.2. Once the system is ON, it stays on with probability 0.1. Find the state transition matrix P.

2. Each second, a laptop computer's wireless LAN card reports the state of the radio channel to an access point. The channel may be (0) poor, (1) fair, (2) good, or (3) excellent. In the poor state, the next state is equally likely to be poor or fair. In states 1, 2, and 3, there is a probability 0.9 that the next system state will be unchanged from the previous state and a probability 0.04 that the next system state will be poor. In states 1 and 2, there is a probability 0.06 that the next state is one step up in quality. When the channel is excellent, the next state is either good with probability 0.04 or fair with probability 0.02. Find the state transition matrix P.

3. The state of a discrete-time Markov chain with transition matrix P can change once each second; X_n denotes the system state after n seconds. An observer examines the system state every m seconds, producing the observation sequence $\hat{X}_0, \hat{X}_1, \ldots$ where $\hat{X}_n = X_{mn}$. Is $\hat{X}_0, \hat{X}_1, \ldots$ a Markov chain? If so, find the state transition matrix $\hat{P}$.

4. The two-state Markov chain can be used to model a wide variety of systems that alternate between ON and OFF states. After each unit of time in the OFF state, the system turns ON with probability p. After each unit of time in the ON state, the system turns OFF with probability q. Using 0 and 1 to denote the OFF and ON states, find a two state transition matrix. What is the probability the system is OFF at time $n = 33$?
Solution:
$$OFF - 0, \qquad ON - 1$$
The state transition matrix is
$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}.$$
Characteristic polynomial is
$$|\lambda I - P| = \begin{vmatrix} \lambda - 1 + p & -p \\ -q & \lambda - 1 + q \end{vmatrix} = \lambda^2 - \lambda(2 - p - q) + 1 - p - q.$$
$$\lambda_1 = \frac{2 - p - q + (p+q)}{2} = 1, \quad \lambda_2 = 1 - (p+q)$$
So, the eigenvalues of P are $\lambda_1 = 1$ and $\lambda_2 = 1 - (p+q)$. Since p and q are probabilities, we have $|\lambda_2| \leq 1$. We can express P in the diagonalized form
$$P = S^{-1}DS = \begin{bmatrix} 1 & -\frac{p}{p+q} \\ 1 & \frac{q}{p+q} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ -1 & 1 \end{bmatrix}.$$
Note that s_i, the $i-$th row of S, is the left eigenvector of P corresponding to λ_i, that is $s_i P = \lambda_i s_i$.

Remark 1. *The same can be done with with the right eigenvectors of P corresponding to λ_i. Then $P = TDT^{-1}$. The $i-$th column of T is the $i-$th right eigenvector, r_i.*
$$\begin{bmatrix} p & -p \\ -q & q \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = y \Rightarrow r_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} q & p \\ q & p \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = -\frac{p}{q}y \Rightarrow r_2 = \begin{bmatrix} -\frac{p}{p+q} \\ \frac{q}{p+q} \end{bmatrix}$$

$$S^{-1} = T = \begin{bmatrix} 1 & -\frac{p}{p+q} \\ 1 & \frac{q}{p+q} \end{bmatrix}, \qquad S = T^{-1} = \begin{bmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ -1 & 1 \end{bmatrix}, \qquad P = TST^{-1}.$$

The $n-$step transition matrix is

$$P^n = \begin{bmatrix} p_{00}(n) & p_{01}(n) \\ p_{10}(n) & p_{11}(n) \end{bmatrix} = S^{-1}D^nS = \frac{1}{p+q}\begin{bmatrix} q & p \\ q & p \end{bmatrix} + \frac{\lambda_2^n}{p+q}\begin{bmatrix} p & -p \\ -q & q \end{bmatrix}.$$

Given the system is OFF at time 0, the conditional probability the system is OFF at time $n = 33$ is

$$p_{00}(33) = \frac{q}{p+q} + \frac{\lambda_2^{33}p}{p+q} = \frac{q + (1-(p+q))^{33}p}{p+q}.$$

5. A packet voice communications system transmits digitized speech only during "talkspurts" when the speaker is talking. In every $10-$ms interval (referred to as a timeslot) the system decides whether the speaker is talking or silent. When the speaker is talking, a speech packet is generated; otherwise no packet is generated. If the speaker is silent in a slot, then the speaker is talking in the next slot with probability $p = 1/140$. If the speaker is talking in a slot, the speaker is silent in the next slot with probability $q = 1/100$. If states 0 and 1 represent silent and talking, find the state transition matrix P for this packet voice system. What is the limiting state probability vector $[p_0^* \;\; p_1^*]$?
Solution: One-step transition probability matrix P is

$$P = \begin{bmatrix} 139/140 & 1/140 \\ 1/100 & 99/100 \end{bmatrix}.$$

The Markov chain is ergodic. Denote by

$$\mathbf{p}(0) = [p_0 \;\; p_1]$$

the initial state probability vector. Then we have

$$\mathbf{p}(n) = \mathbf{p}(0)P^n.$$

We can solve this problem on two different ways:

(a) In the previous problem we have found P^n. Here $p = 1/140$ and $q = 1/100$. Also

$$\begin{aligned}\mathbf{p}(n) &= [p_0(n) \;\; p_1(n)] \\ &= \left[\frac{q}{p+q}(p_0+p_1) \quad \frac{p}{p+q}(p_0+p_1)\right] + \lambda_2^n\left[\frac{p_0p - p_1q}{p+q} \quad \frac{-p_0p + p_1q}{p+q}\right],\end{aligned}$$

where $\lambda_2 = 1 - (p+q) = 1 - \dfrac{5+7}{700} = \dfrac{344}{350}$ and $p_0 + p_1 = 1$. So,

$$\mathbf{p}(n) = \left[\frac{7}{12} \;\; \frac{5}{12}\right] + \left(\frac{344}{350}\right)^n\left[\frac{5}{12}p_0 - \frac{7}{12}p_1 \quad -\frac{5}{12}p_0 + \frac{7}{12}p_1\right].$$

Since $|\lambda_2| < 1$, the limiting state probabilities are

$$\lim_{n\to\infty}\mathbf{p}(n) = \left[\frac{7}{12} \;\; \frac{5}{12}\right].$$

For this system, the limiting state probabilities are the same regardles of how we chose the initial state probabilities.

(b) Since the Markov chain is ergodic we have

$$[p_1^* \quad p_2^*] = [p_1^* \quad p_2^*] \begin{bmatrix} 139/140 & 1/140 \\ 1/100 & 99/100 \end{bmatrix},$$
$$p_1^* + p_2^* = 1.$$

$$\begin{aligned} p_1^* &= \frac{139}{140}p_1^* + \frac{1}{140}p_2^*, \\ p_2^* &= \frac{1}{140}p_1^* + \frac{99}{140}p_2^*, \\ p_1^* + p_2^* &= 1. \end{aligned}$$

From the first equation we have $p_1^* = \frac{7}{5}p_2^*$, and from the third we get

$$p_1^* = \frac{5}{12}, \quad p_2^* = \frac{7}{12}.$$

6. Each of two switches is either on or off during a day. On day n, each switch will independently be off with probability

$$\frac{1 + \text{ number of on switches during day } (n-1)}{4}.$$

For instance, if both switches are on during day $n-1$, then each will independently be off during day n with probability 3/4. What fraction of days are both switches on? What fraction are both off?

7. A DNA nucleotide has any of 4 values. A standard model for a mutational change of the nucleotide at a specific location is a Markov chain model that supposes that in going from period to period the nucleotide does not change with probability $1-3\alpha$, and if it does change then it is equally likely to change to any of the other 3 values, for some $0 < \alpha < \frac{1}{3}$.

   (a) Show that $p_{11}(n) = \dfrac{1}{4} + \dfrac{3}{4}(1-4\alpha)^n$.

   (b) What is the long run proportion of time the chain is in each state?

8. Ana is doing data analysis and she is receiving data through a communication system. But, she is aware that digit entered might be changed in the process and based of the prediction analysis she knows which digit is unchanged/changed. Previously, she received a transition probability matrix P for a three state Markov chain

$$P = \begin{bmatrix} a & b & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}.$$

Ana knows that, in the long run, 25% of the time system spends in the state 2. Find constants a and b, if possible.

9. The diffusion of electrons and holes across a potential barrier in an electronic devise is modeled as follows. There are m black balls (electrons) in urn A and m white balls (holes) in urn B. We perform independent trials, in each of which a ball is selected at random from each urn and the selected ball from urn A is placed in urn B, while that from urn B is placed in A. Consider the Markov chain representing the number of black balls in urn A immediately after the $n-$th trial.

(*a*) Describe the one-step transition probabilities of the process.

(*b*) Suppose $m = 2$. Compute the long-run fraction of time when urn A does not contain a black ball.

10. In working with a particular gene for fruit flies, geneticists classify an individual fruit fly as dominant, hybrid or recessive. In running an experiment, an individual fruit fly is crossed with a hybrid, then the offspring is crossed with a hybrid and so forth. The offspring in each generation are recorded as dominant, hybrid or recessive. The probabilities the offspring are dominant, hybrid or recessive depends only on the type of fruit fly the hybrid is crossed with rather than the genetic makeup of previous generations. The offspring of a dominant individual crossed with a hybrid are dominant 50% of the time and hybrid the other 50%. The offspring of a hybrid crossed with a hybrid are dominant 25%, hybrid 50% and recessive 25%, while the offspring of a recessive crossed with a hybrid are hybrid 50% and recessive 50%.

    (*a*) Find the transition matrix for this problem.

    (*b*) What is the probability the third generation offspring is dominant given the first generation offspring is recessive?

    (*c*) If the population of fruit flies initially is 20% dominant, 50% hybrid and 30% recessive, what percentage of the population is dominant after 3 generations?

11. The social status of the $n-$th generation of some family is given by the Markov chain $X_n, n \in \mathbb{N}_0$. Some family can belong to one of three social classes: 1 - lower, 2 - middle and 3 - upper. The transition probability matrix which describes the change in classes is given by
$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}.$$

    (*a*) Find a probability that a person whose parents are in the middle class ends up in the upper class and that the children of that person are in the lower class.

    (*b*) Determine a probability that the children of the person whose parents are in the middle class end up in the lower class?

12. In some countries exists a limit on number of children the couple can have in marriage. Suppose that a limit is 4, i.e. a couple cannot have more that 4 children in marriage. Let X_n denote the number of children in marriage after n years. Suppose that the number of children cannot decrease. It is known that the number of children after one year will stay the same with probability 0.5, except in the case when there is 4 children. If in the end of one year the number of children is k, $k = 0, 1, 2$, then at the end of the next year the number of children will be $k + 1$ with probability $\frac{1}{k+2}$. If the couple has a child, the number of children will increase by 2 after 2 year with probability 1/4.

    (*a*) Find a one-step transition probability matrix of a Markov chain $\{X_n\}$.

    (*b*) Is Markov chain $\{X_n\}$ ergodic?

Eigenvalues and eigenvectors of a matrix

For a given square matrix A a **characteristic polynomial** is $p_A(\lambda) = \det(\lambda I - A)$, where I is identity matrix. The spectrum of a matrix A, $\sigma(A) = \{\lambda \in \mathbb{C} : p_A(\lambda) = 0\}$ is the set of its eigenvalues i.e. $\lambda \in \mathbb{C}$ is an eigenvalue of a matrix A is and only if $\lambda \in \sigma(A)$. Vector $x \in \ker(\lambda I - A)$ is an eigenvector corresponding to the eigenvalue λ. The **right eigenvector** is a column vector satisfying $\lambda x = Ax$ and the **left eigenvector** is a row vector satisfying $\lambda x = xA$.
Cayley Hamilton theorem: $p_A(A) = 0$.

The eigendecomposition (or spectral decomposition) of a diagonalizable matrix A is a decomposition of a diagonalizable matrix into a specific canonical form whereby the matrix is represented in terms of its eigenvalues and eigenvectors. Suppose that the eigenvalues of a $n \times n$ matrix A are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$. The right eigenvector corresponding to λ_i is denoted by r_i and the left eigenvector corresponding to λ_i is denoted by l_i. So,

$$\lambda_i r_i = A r_i, \quad \lambda_i l_i = l_i A.$$

It holds

$$\det(A) = \lambda_1 \lambda_2 \cdot \ldots \cdot \lambda_n, \quad \mathrm{tr}(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n.$$

Then

$$A = S^{-1} D S = T D T^{-1},$$

where $D = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, the $i-$th row of a matrix S is l_i, while the $i-$th column of a matrix T is r_i. Then

$$A^n = S^{-1} D^n S = T D^n T^{-1}.$$