Martingales

Definition 1. *A sequence $\xi_1, \xi_2, \ldots$ of random variables is called a* **martingale** *with respect to a filtration $\mathcal{F}_1, \mathcal{F}_2, \ldots$ if*

1. *ξ_n is integrable for each $n = 1, 2, \ldots$,*
2. *$\xi_1, \xi_n, \ldots$ is adapted to $\mathcal{F}_1, \mathcal{F}_2, \ldots$,*
3. *$E(\xi_{n+1}|\mathcal{F}_n) = \xi_n$ a.s. for each $n = 1, 2, \ldots$.*

Example 1. If $\mathcal{F}_n = \mathcal{F}(\xi_1, \xi_2, \ldots, \xi_n)$ is $\sigma-$field generated by $\xi_1, \xi_2, \ldots, \xi_n$ then $\xi_1, \xi_2, \ldots$ is adapted to $\mathcal{F}_1, \mathcal{F}_2, \ldots$.

Problem 1. Let $\eta_1, \eta_2, \ldots$ be a sequence of independent integrable random variables such that $E(\eta_n) = 0$ for all $n = 1, 2, \ldots$. Let

$$\xi_n = \eta_1 + \ldots + \eta_n,$$
$$\mathcal{F}_n = \mathcal{F}(\eta_1, \ldots, \eta_n).$$

Show that a sequence $\xi_1, \xi_2, \ldots$ is a martingale with respect to a filtration $\mathcal{F}_1, \mathcal{F}_2, \ldots$.

Problem 2. Show that if $\xi_1, \xi_2, \ldots$ is a martingale with respect to $\mathcal{F}_1, \mathcal{F}_2, \ldots$ then $E(\xi_1) = E(\xi_2) = \ldots$.

Problem 3. Let $\{N(t),\, t \geq 0\}$ be a Poisson process with rate λ. Show that the process $N(t) - \lambda t$ is martingale.

Problem 4. Let ξ_n be a symmetric random walk, that is,

$$\xi_n = \eta_1 + \ldots + \eta_n,$$

where $\eta_1, \eta_2, \ldots$ is a sequence of independent identically distributed random variables such that

$$P(\eta_n = 1) = P(\eta_n = -1) = \frac{1}{2}.$$

Show that $\xi_n^2 - n$ is a martingale with respect to the filtration $\mathcal{F}_n = \mathcal{F}(\eta_1, \ldots, \eta_n)$.

Problem 5. Let ξ_n be a symmetric random walk, $\xi_n = \eta_1 + \ldots + \eta_n$, where $\eta_1, \eta_2, \ldots$ is a sequence of independent identically distributed random variables such that

$$P(\eta_n = 1) = P(\eta_n = -1) = \frac{1}{2}$$

and $\mathcal{F}_n$ the filtration defined by $\mathcal{F}_n = \mathcal{F}(\eta_1, \ldots, \eta_n)$. Show that

$$\gamma_n = (-1)^n \cos(\pi \xi_n)$$

is martingale with respect to $\mathcal{F}_n$.

Problem 6. Let ξ_n be a sequence of square integrable random variables. Show that if ξ_n is a martingale with respect to a filtration $\mathcal{F}_n$, then ξ_n^2 is a submartingale with respect to the same filtration.

Example 2. Suppose that you take part in a game such as the roulette, for example. Let $\eta_1, \eta_2, \ldots$ be a sequence of integrable random variables, where η_n are your winnings (or losses) per unit stake in game n. If your stake in each game is one, then your total winnings after n games will be

$$\xi_n = \eta_1 + \ldots + \eta_n$$

We take the filtration

$$\mathcal{F}_n = \mathcal{F}(\eta_1, \ldots, \eta_n)$$

and also put $\xi_0 = 0$.
If $n-1$ rounds of the game have been played so far, your accumulated knowledge will be represented by the $\sigma-$field $\mathcal{F}_{n-1}$. The game is fair if

$$E\big(\xi_n | \mathcal{F}_{n-1}\big) = \xi_{n-1},$$

that is, you expect that your fortune at step n will on average be the same as at step $n-1$. The game will be favourable to you if

$$E\big(\xi_n | \mathcal{F}_{n-1}\big) \geq \xi_{n-1},$$

and unfavourable to you if

$$E\big(\xi_n | \mathcal{F}_{n-1}\big) \leq \xi_{n-1},$$

for $n = 1, 2, \ldots$. This corresponds to ξ_n being, respectively, a martingale, a submartingale, or a supermartingale with respect to $\mathcal{F}_n$.

Martingales and Brownian motion W_t

Let $\{W_t,\, t \geq 0\}$ be a Brownian motion process. Denote by

$$\mathcal{W}_s = \mathcal{F}(W_t,\, 0 \leq t \leq s)$$

the history of Brownian motion until time s.

Theorem 1. *W_t is martingale with respect to $\mathcal{W}_s$.*

Theorem 2. *$W_t^2 - t$ is martingale with respect to the history of Brownian motion $\mathcal{W}_s$.*