Brownian motion

Definition 1. *A stochastic process* $\{W_t, t \geq 0\}$ *is said to be (standard) Brownian motion (Wiener process) process if*

(i) $W_0 = 0$

(ii) $\{W_t, t \geq 0\}$ *has stationary and independent increments i.e. for each* $0 < t_1 < t_2 < \ldots < t_{n-1} < t_n < \ldots$, $W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$ *are independent random variables.*

(iii) $W_t - W_s : \mathcal{N}(0, t-s)$, $t > s$. *Specially, if we take* $s = 0$ *we have*

$$W_t - W_0 = W_t : \mathcal{N}(0, t), \quad E[W_t] = 0, \quad D[W_t] = E[W_t^2] = t.$$

Problem 1. Let $\{W_t,\, t \geq 0\}$ be a Brownian motion.

(a) Find $P\{1 < W_1 < 2\}$.

(b) Find $P\{W_2 < 3 | W_1 = 1\}$.

Problem 2. Over the course of a day, the stock price of a widely traded company can be modeled as a Brownian motion process where $X(0)$ is the opening price at the morning bell. Suppose the unit of time t is an hour, the exchange is open for eight hours, and the standard deviation of the daily price change (the difference between the opening bell and closing bell prices) is 1/2 point. What is the value of the Brownian motion parameter α?

Problem 3. Let $X(t)$ be a Brownian motion process with variance $Var[X(t)] = \alpha t$. For a constant $c > 0$, determine whether $Y(t) = X(ct)$ is a Brownian motion process.

Problem 4. If $\{W_t,\, t \geq 0\}$ ia a Brownian motion process with drift coefficient η and variance parameter σ^2, then the process $\{X_t,\, t \geq 0\}$ defined by

$$X_t = e^{W_t}$$

is called *geometric Brownian motion.* Prove that

$$E[X_t] = e^{\eta t + \frac{\sigma^2 t}{2}}.$$