Random variables

Problem 1. A manufacturer of a consumer electronics product expects 2% of units to fail during the warranty period. A sample of 500 independent units is tracked for warranty performance.

(a) What is the probability that none fails during the warranty period?

(b) What is the expected number of failures during the warranty period?

(c) What is the probability that more than two units fail during the warranty period?

Problem 2. An installation technician for a specialized communication system is dispatched to a city only when three or more orders have been placed. Suppose orders follow a Poisson distribution with a mean of 0.25 per week for a city with a population of 100 000 and suppose your city contains a population of 800 000.

(a) What is the probability that a technician is required after a one-week period?

(b) If you are the first one in the city to place an order, what is the probability that you have to wait more than two weeks from the time you place your order until a technician is dispatched?

Example 1. Consider the transmission of n bits over a digital communication channel. Let the random variable X equal the number of bits in error. When the probability that a bit is in error is constant and the transmissions are independent, X has a binomial distribution. Let p denote the probability that a bit is in error. Let $\lambda = pn$. The $E(X) = pn = \lambda$ and

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}.$$

Now, suppose that the number of bits transmitted increases and the probability of an error decreases exactly enough that pn remains equal to a constant. That is, n increases and p decreases accordingly, such that $E(X) = \lambda$ remains constant. Then, with some work, it can be shown that

$$\lim_{n\to\infty} P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \qquad x = 0, 1, 2, \ldots.$$

Also, because the number of bits transmitted tends to infinity, the number of errors can equal any nonnegative integer. Therefore, the range of X is the integers from zero to infinity.

Problem 3. An airline knows that 5 percent of the people making reservations on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for a flight that can hold only 50 passengers. What is the probability that there will be a seat available for every passenger who shows up?

Problem 4. When someone presses "SEND" on a cellular phone, the phone attempts to set up a call by transmitting a "SETUP" message to a nearby base station. The phone waits for a response and if none arrives within 0.5 seconds it tries again. If it doesn't get a response after $n = 6$ tries the phone stops transmitting messages and generates a busy signal.

(a) Draw a tree diagram that describes the call setup procedure.

(b) If all transmissions are independent and the probability is p that a "SETUP" message will get through, what is the PMF of K, the number of messages transmitted in a call attempt?

(c) What is the probability that the phone will generate a busy signal?

(d) As manager of a cellular phone system, you want the probability of a busy signal to be less than 0.02. What is the minimum value of p necessary to achieve your goal?

Problem 5. In the development of a new receiver for the transmission of digital information, each received bit is rated as acceptable, suspect, or unacceptable, depending on the quality of the received signal, with probabilities 0.9, 0.08, and 0.02, respectively. Assume that the ratings of each bit are independent. In the first four bits transmitted, let

X denote the number of acceptable bits
Y denote the number of suspect bits

Determine joint probability distribution of X and Y $p_{X,Y}(x, y)$, marginal probability distribution of X, $E(X)$, $P(Y = 0|X = 3)$ and $P(Y = 1|X = 3)$.

Problem 6. A small-business Web site contains 100 pages and 60%, 30% and 10% of the pages contain low, moderate and high graphic content, respectively. A sample of four pages is selected without replacement and X and Y denote the number of pages with moderate and high graphics output in the sample. Determine

(a) $p_{X,Y}(x, y)$

(b) $p_X(x)$

(c) $E(X)$

(d) $p_{Y|X=3}(y)$

(e) $E(Y|X = 3)$

(f) Are X and Y independent?

Problem 7. [Piecewice Constant PDF] Alvin's driving time to work is between 15 and 20 minutes if the day is sunny and between 20 and 25 minutes if the day is rainy, with all times being equally likely in each case. Assume that a day is sunny with probability 2/3 and rainy with probability 1/3. What is the PDF of the driving time, viewed as a random variable X?

Problem 8. Let the continuous random variable X denote the diameter of a hole drilled in a sheet metal component. The target diameter is 12.5 millimeters. Most random disturbances to the process result in larger diameters. Historical data show that the distribution of X can be modeled by a probability density function $\varphi(x) = 20e^{-20(x-12.5)}$, $x \geq 12.5$. If a part with a diameter larger than 12.60 millimeters is scrapped, what proportion of parts is scrapped? What proportion of parts is between 12.5 and 12.6 millimeters?

The Normal Random Variable $\mathcal{N}(\mu, \sigma^2)$

For $\sigma > 0$, $-\infty < \mu < \infty$

$$\varphi_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad E(X) = \mu, \qquad \mathrm{var}(X) = \sigma^2.$$

Special case: **Standard Normal random variable $\mathcal{N}(0, 1)$:**

$$\varphi_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \qquad \Phi(x) = \int_0^x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx.$$

1. $\Phi(0) = 0$
2. $\Phi(-x) = -\Phi(x)$
3. $\Phi(x) = \frac{1}{2}$, for $x \geq 5$
4. $\Phi(x) \approx \frac{1}{2}$ for $3.5 \leq x \leq 5$

Problem 9. [Signal detection] A binary message is transmitted as a signal s, which is either -1 or +1. The communication channel is transmitted with additive normal noise with mean $\mu = 0$ and variance σ^2. The receiver concludes that the signal -1 (or +1) was transmitted if the value received is < 0 (or ≥ 0, respectively). What is the probability of error?

Problem 10. Suppose the time it takes a data collection operator to fill out an electronic form for a database is uniformly between 1.5 and 2.2 minutes.

(a) What is the mean and variance of the time it takes an operator to fill out the form?

(b) What is the probability that it will take less than two minutes to fill out the form?

(c) Determine the cumulative distribution function of the time it takes to fill out the form.

Problem 11. Let X be uniformly distributed over $(0, 1)$. Calculate $E[X^3]$.

Problem 12. Random variables X and Y have joint PDF

$$\varphi_{X,Y}(x, y) = \begin{cases} c, & 0 \leq x \leq 5,\ 0 \leq y \leq 3 \\ 0, & otherwise \end{cases}$$

Find the constant c, $P\{2 \leq X < 3, 1 \leq Y < 2\}$ and $P\{Y > X\}$.

Problem 13. The cumulative distribution function of random variable U is

$$F_U(u) = \begin{cases} 0, & u < -5 \\ \frac{u+5}{8}, & -5 \leq u < -3 \\ \frac{1}{4}, & -3 \leq u < 3 \\ \frac{1}{4} + \frac{3(u-3)}{8}, & 3 \leq u < 5 \\ 1, & u \geq 5 \end{cases}.$$

(a) What is $E(U)$?

(b) What is $\mathrm{var}(U)$?

(c) What is $E(2^U)$?

Problem 14. [The Exponential Random Variable is Memoryless] The time T until a new light bulb burns out is an exponential random variable with parameter λ. Ariadne turns the light on, leaves the room and when she returns, t time units later, finds that the light bulb is still on, which corresponds to the event $A = \{T > t\}$. Let X be the additional time until the light bulb burns out. What is the conditional CDF of X, given the event A?

Theorem 1 (Central Limit Theorem). *Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables, each with mean μ and variance σ^2. Then the distribution of*

$$\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

tends to the standard normal as $n \to \infty$.

Problem 15. The lifetime of a special type of battery is a random variable with mean 40 hours and standard deviation 20 hours. A battery is used until it fails, at which point it is replaced by a new one. Assuming a stockpile of 25 such batteries, the lifetimes of which are independent, approximate the probability that over 1100 hours of use can be obtained.

Theorem 2 (De Moivre-Laplace theorem). *Consider a sequence of Bernoulli trials with probability p of success. Let S_n, $n \in \mathbb{N}$, denote the number of successes in the first n trials. For any $a, b \in \mathbb{R}$, with $a < b$,*

$$\lim_{n\to\infty} P\Big\{a < \frac{S_n - np}{\sqrt{npq}} < b\Big\} = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}x^2}\,dx = \Phi(b) - \Phi(a).$$

De Moivre-Laplace theorem is a special case of the Central Limit Theorem. It is useful in the cases when one has the binomial random variable with n large. More precisely, when n is large and $np \geq 10$ binomial random variable can be approximated by normal random variable.

Problem 16. A certain population is comprised of half men and half women. In a random sample of 10 000 what is the chance that the percentage of the men in the sample is somewhere between 49% and 51%?