Conditional expectation

Problem 1. Sam should read one chapter of his probability book and one chapter of his history book but he is having problem deciding which one to read the first. Then he decides to toss the fair coin twice. If the number of heads is even he will read his probability book first. Let X be the number of heads and let A be the event that the number of heads is even. Find $E(X|A)$ and $E(X|\mathcal{F}(A))$, where $\mathcal{F}(A) = \{\emptyset, A, \bar{A}, \Omega\}$.

Problem 2. Suppose that the random variable X has the piecewise constant PDF

$$\varphi_X(x) = \begin{cases} 1/3, & \text{if } 0 \le x \le 1, \\ 2/3, & \text{if } 1 < x \le 2, \\ 0, & \text{otherwice} \end{cases}$$

Consider the events

$$A_1 = \{X \text{ lies in the first interval } [0,1]\},$$
$$A_2 = \{X \text{ lies in the second interval } (1,2]\}.$$

Find $E(X|\mathcal{F}(A_1, A_2))$, $E(X)$ and $\text{var}(X)$.

Problem 3. A coin, having probability p of coming up heads, is to be successively flipped until the first head appears. Let N be the number of flips required, and let A be the event that the first flip results in a head. Find $E(N|\mathcal{F}(A))$ and the expected number of flips required.

Problem 4. A miner is trapped in a mine containing three doors. The first door leads to a tunnel that takes him to safety after two hours of travel. The second door leads to a tunnel that returns him to the mine after three hours of travel. The third door leads to a tunnel that returns him to his mine after five hours. Assume that the miner is at all times equally likely to choose any one of the doors, and let A_i be the event that a miner chooses the door number i, $i = 1, 2, 3$. If X denotes the time until the miner reaches safety, find $E\big(X|\mathcal{F}(A_1, A_2, A_3)\big)$ and the expected length of time until the miner reaches safety?

Problem 5. Consider n independent random variables X_1, $X_2, \ldots, X_n$, each with probability distribution

$$X_i : \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}, \; i = 1, \ldots, n.$$

Let $S_n = \sum_{k=1}^{n} X_k$. Find

1. PMF for S_n.
2. PMF for $S_n|\{X_n = 0\}$ and $S_n|\{X_n = 1\}$
3. PMF for $X_n|\{S_n = k\}$, $k = 0, 1, \ldots, n$ and $E(X_n|S_n = k)$, $k = 0, \ldots, n$.
4. $E(X_n|S_n)$

5. Show that $E(E(X_n|S_n)) = E(X_n)$ holds true.

Problem 6. Let X be the randomly chosen number from the set $\{1, 2, 3, 4\}$ and let Y be the randomly chosen number from the same set but not greater than X. Find $E(X|Y)$.

Problem 7. A and B play a series of games with A winning each game with probability p. The overall winner is the first player to have won two more games than the other.

(a) Find the probability that A is the overall winner.

(b) Find the expected number of games played.

Problem 8. The taxi stand and the bus stop near Al's home are in the same location. Al goes there at a given time and if a taxi is waiting (this happens with probability 2/3) he board it. Otherwise he waits for a taxi or a bus to come, whichever comes first. The next taxi will arrive in a time that is uniformly distributed between 0 and 10 minutes, while the next bus will arrive in exactly 5 minutes. Find the CDF and the expected values of Al's waiting time.