Martingali i Braunovo kretanje

1. Let $\eta_1, \eta_2, \ldots$ be a sequence of independent integrable random variables such that $E(\eta_n) = 0$ for all $n = 1, 2, \ldots$. Let

$$\xi_n = \eta_1 + \ldots + \eta_n,$$
$$\mathcal{F}_n = \mathcal{F}(\eta_1, \ldots, \eta_n).$$

   Show that a sequence $\xi_1, \xi_2, \ldots$ is a martingale with respect to a filtration $\mathcal{F}_1, \mathcal{F}_2, \ldots$.

2. Show that if $\xi_1, \xi_2, \ldots$ is a martingale with respect to $\mathcal{F}_1, \mathcal{F}_2, \ldots$ then $E(\xi_1) = E(\xi_2) = \ldots$.

3. Let ξ_n be a symmetric random walk, $\xi_n = \eta_1 + \ldots + \eta_n$, where $\eta_1, \eta_2, \ldots$ is a sequence of independent identically distributed random variables such that

$$P(\eta_n = 1) = P(\eta_n = -1) = \frac{1}{2}$$

   and $\mathcal{F}_n$ the filtration defined by $\mathcal{F}_n = \mathcal{F}(\eta_1, \ldots, \eta_n)$. Show that

$$\gamma_n = (-1)^n \cos(\pi \xi_n)$$

   is martingale with respect to $\mathcal{F}_n$.

4. Show that $e^{W_t} e^{-t/2}$ is a martingale if W_t is standard Brownian motion.

5. Show that for any $T > 0$, $V(t) = W(t+T) - W(T)$ is a standard Brownian motion if $W(t)$ is.

6. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables, such that $P\{X_n = 1\} = p$ and $P\{X_n = -1\} = q$ for all $n \in \mathbb{N}$ and $0 < p < 1, q = 1 - p$. Define $S_n := \sum_{j=1}^n X_j,\ n \geq 1$.

   (a) Compute the probability mass function of the random variable $Y_i := \left(\frac{q}{p}\right)^{X_i}$, $i \in \mathbb{N}$.

   (b) Prove that the sequence of random variables

$$M_n := \left(\frac{q}{p}\right)^{S_n}$$

   is a martingale with respect to a filtration $\mathcal{F}_n = \sigma(X_j, 1 \leq j \leq n),\ n \geq 1$. With $\sigma(X_j, 1 \leq j \leq n),\ n \geq 1$ we denote σ-algebra generated by random variables $X_j, 1 \leq j \leq n$.

7. Let $(W_t)_{t\geq 0}$ be a standard Brownian motion. Find $E(W_4^2 | W_2)$.

8. $X_1, X_2, \ldots$ is a sequence of independent identically distributed random variables with zero expectation and finite variance. Suppose that X_i cannot take absolute values greater than 1. Define

$$S_n := X_1 + \ldots + X_n, \quad n \in \mathbb{N}.$$

   Find a constant a such that $S_n^2 - aE(S_n^2)$ is a martingale with respect to a filtration $\mathcal{F}_n$ which is a $\sigma-$algebra generated by $X_1, \ldots, X_n$.

9. Consider a function $f : [0, \infty) \to \mathbb{R}$ and define a process

$$X_t := W_t^3 + f(t)W_t, \ t \geq 0,$$

where W_t is a standard Brownian motion. Determine a function f such that $\{X_t\}$ is a martingale.

10. Consider a standard Brownian motion W_t and define

$$X_t := W_t - tW_1, \ 0 \leq t \leq 1.$$

(a) Find a autocovariance function $K_X(t, s)$ for a given process X_t.

(b) Does the process $\{X_t\}_{0 \leq t \leq 1}$ has an independent increments?

11. Consider the standard Brownian motion $W(t)$, $t \geq 0$.

(a) Prove that for $s \geq t \geq 0$

$$E(W(s)W^n(t)) = \begin{cases} 0, & \text{if } n = 2k, \ k = 0, 1, 2, \dots \\ n!!\, t^{\frac{n+1}{2}}, & \text{if } n = 2k - 1, \ k = 1, 2, \dots \end{cases}.$$

(b) Show that $E\big(|W(t) - W(s)|^2\big) = |t - s|$, $t, s \geq 0$.

Hint: Calculate $E(W^{n+1}(t))$.

12. Find a constant a such that

$$V(t) = W^3(t) + aW^2(t) - 3tW(t) + t, \quad t \geq 0$$

is martingale, if $W(t)$ is standard Brownian motion.