Conditional expectations

1. Sam should read one chapter of his probability book and one chapter of his history book but he is having problem deciding which one to read the first. Then he decides to toss the fair coin twice. If the number of heads is even he will read his probability book first. Let X be the number of heads and let A be the event that the number of heads is even. Find $E(X|A)$ and $E(X|\mathcal{F}(A))$, where $\mathcal{F}(A) = \{\emptyset, A, \bar{A}, \Omega\}$.

2. Suppose that the random variable X has the piecewise constant PDF

$$\varphi_X(x) = \begin{cases} 1/3, & \text{if } 0 \leq x \leq 1, \\ 2/3, & \text{if } 1 < x \leq 2, \\ 0, & \text{otherwice} \end{cases}$$

Consider the events

$$A_1 = \{X \text{ lies in the first interval } [0,1]\},$$
$$A_2 = \{X \text{ lies in the second interval } (1,2]\}.$$

Find $E(X|\mathcal{F}(A_1, A_2))$, $E(X)$ and $\text{var}(X)$.

3. A miner is trapped in a mine containing three doors. The first door leads to a tunnel that takes him to safety after two hours of travel. The second door leads to a tunnel that returns him to the mine after three hours of travel. The third door leads to a tunnel that returns him to his mine after five hours. Assume that the miner is at all times equally likely to choose any one of the doors, and let A_i be the event that a miner chooses the door number i, $i = 1, 2, 3$. If X denotes the time until the miner reaches safety, find $E\big(X|\mathcal{F}(A_1, A_2, A_3)\big)$ and the expected length of time until the miner reaches safety?

4. A and B play a series of games with A winning each game with probability p. The overall winner is the first player to have won two more games than the other.

   (a) Find the probability that A is the overall winner.

   (b) Find the expected number of games played.

5. A prisoner is trapped in a cell containing three doors. The first door leads to a tunnel that returns him to his cell after two days of travel. The second leads to a tunnel that returns him to his cell after three days of travel. The third door leads immediately to freedom.

   (a) Assuming that the prisoner will always select doors 1, 2, and 3 with probabilities 0.5, 0.3, 0.2, what is the expected number of days until he reaches freedom?

   (b) Assuming that the prisoner is always equally likely to choose among those doors that he has not used, what is the expected number of days until he reaches freedom? (In this version, for instance, if the prisoner initially tries door 1, then when he returns to the cell, he will now select only from doors 2 and 3.)

6. Suppose that 2 batteries are randomly chosen without replacement from the following group of 12 batteries: 3 new, 4 used (working) and 5 defective. Let X denote the number of new batteries chosen and let Y denote the number of used batteries chosen. Find $E(X)$ and $E(X|Y)$.

7. Each day a system is gathering a certain amount of data and sums the money earned that day. Then a program classifies the previous day into a category. Each day can be classified as a Type i, $i \in \{1, \ldots, k\}$. It is known that probability the day is classified as Type i is p_i and that

$$\sum_{i=1}^{k} p_i^2 = 0.5.$$

Also, the expected amount of money earned during the Type i day is $\ln\left(\frac{a}{e^{p_i}}\right)$ millions, $a > 1$. What is the expected amount of money earned on arbitrary chosen day?

8. Suppose that a couple after n year of marriage can have at most n children. Let X denote a random variable which represents the number of children after exactly 3 years of marriage,

$$X : \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1/6 & 1/3 & 1/3 & 1/6 \end{pmatrix}.$$

If Y is a number of female children after exactly 3 years of marriage, find $E(X|\mathcal{F}(Y))$.

9. The taxi stand and the bus stop near Al's home are in the same location. Al goes there at a given time and if a taxi is waiting (this happens with probability 2/3) he board it. Otherwise he waits for a taxi or a bus to come, whichever comes first. The next taxi will arrive in a time that is uniformly distributed between 0 and 10 minutes, while the next bus will arrive in exactly 5 minutes. Find the CDF and the expected values of Al's waiting time.