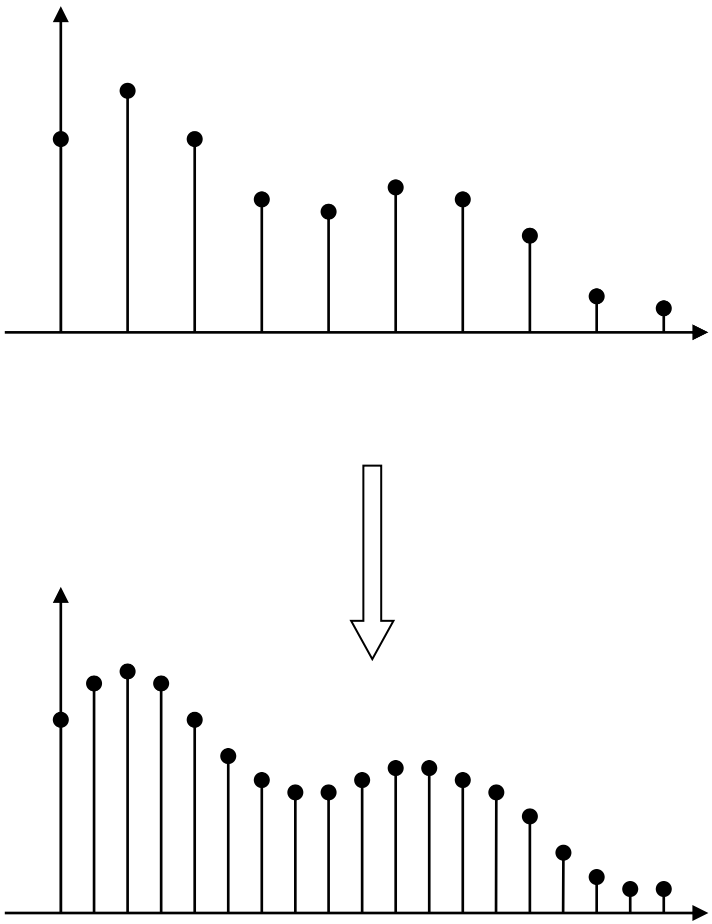
MULTIRATE SYSTEMS

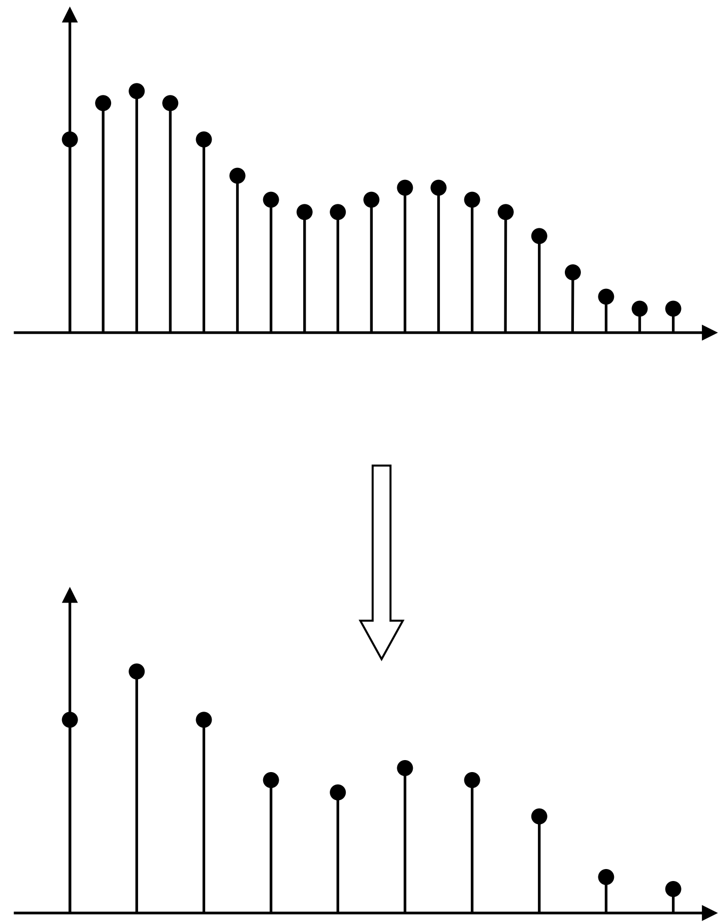
Multirate signal processing

- Systems sometimes have to work with signals sampled at different sampling frequencies (sampling rates)
- Sampling rate conversion
  - Via continuous-time domain
  - Completely in discrete-time domain
    - Based on *interpolation* and *decimation*
- Applications
  - Interface between systems working at different sampling frequencies
  - Oversampling (to simplify anti-aliasing filter)
  - Efficient implementation of some DSP algorithms
    - Narrowband FIR filtering

Interpolation

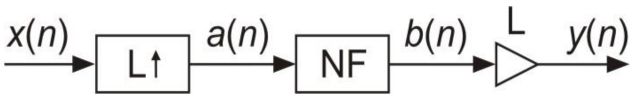


Decimation



Interpolation

- Increasing the sampling frequency L times in the digital domain



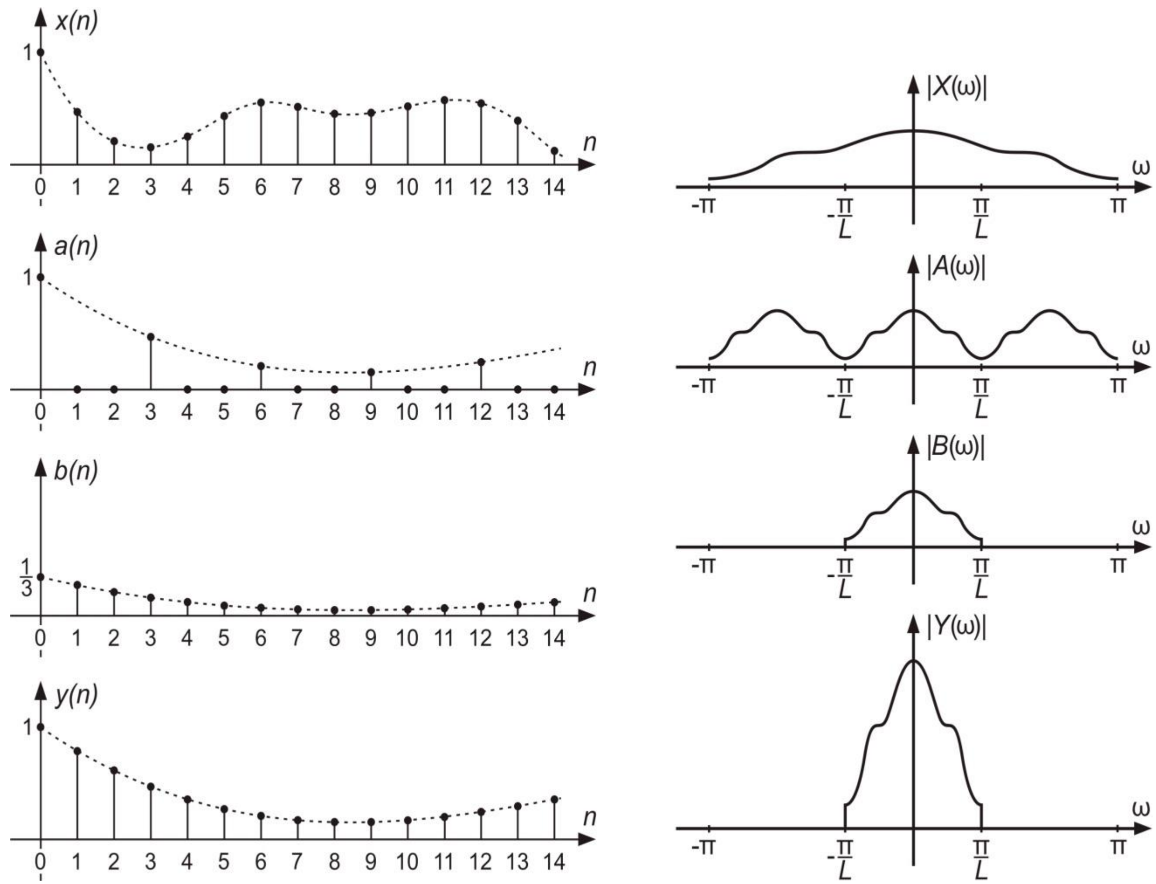
$$a(n)=\begin{cases} x(n/L), & n=0,\pm L,\pm 2L\ldots \\ 0, & drugde \end{cases}$$

$$A(z)=\sum_{n=-\infty}^{\infty} a(n)z^{-n}=\sum_{n=-\infty}^{\infty} x(n)z^{-nL}$$

$$A(\omega)=X(L\omega)$$

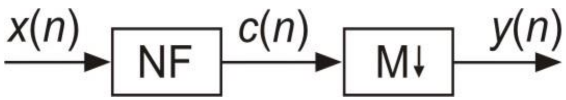
$$Y(\omega)=L\cdot H_{NF}(\omega)A(\omega)$$

$$=\begin{cases} L\cdot X(L\omega), & |\omega|\le \pi/L \\ 0, & |\omega|>\pi/L \end{cases}$$



Decimation

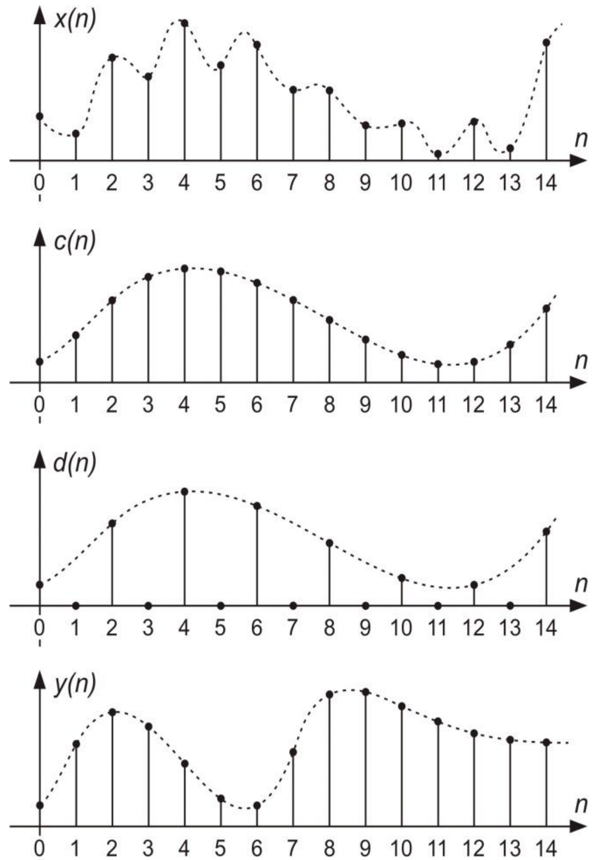
- Decreasing the sampling frequency *M* times in the digital domain

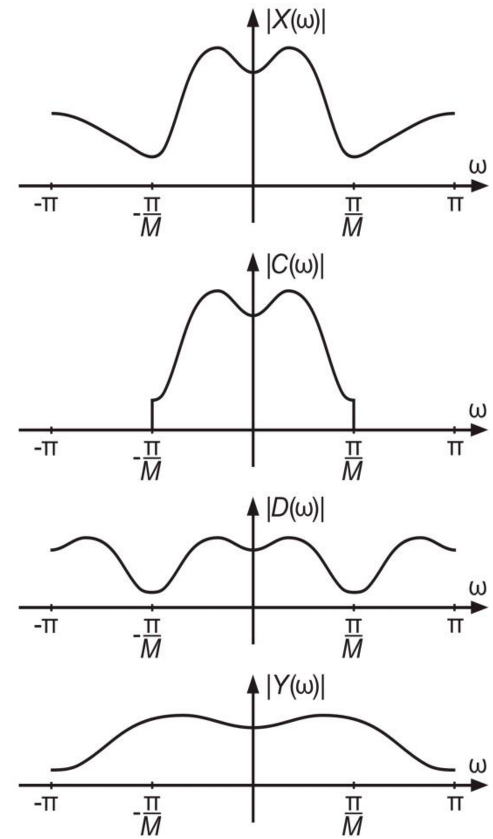


$$y(n) = c(Mn)$$

$$d(n) = \begin{cases} c(n), & n = kM \\ 0, & n \neq kM \end{cases}$$

$$= c(n)\delta_M(n)$$





Decimation

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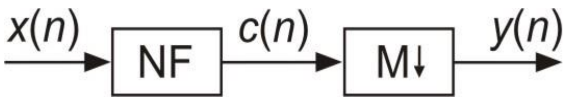


$$Y(z)=\sum_{n=-\infty}^{\infty} y(n)z^{-n}=\sum_{n=-\infty}^{\infty} c(Mn)z^{-n}=\sum_{n=-\infty}^{\infty} d(Mn)z^{-n}$$

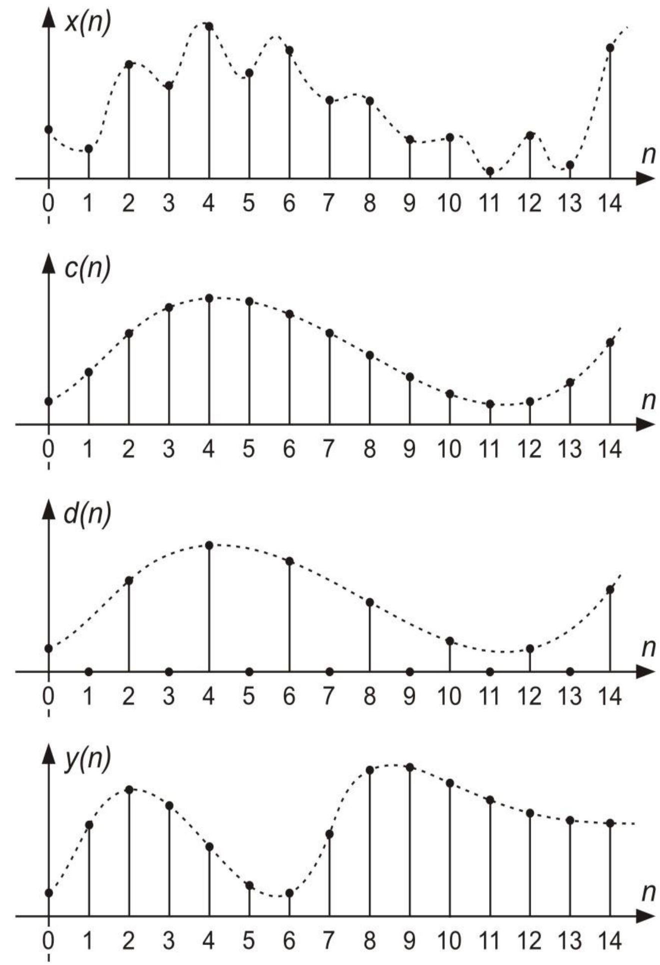
$$=\sum_{\substack{m=-\infty \\ m=kM}}^{\infty} d(m)z^{-m/M}=\sum_{n=-\infty}^{\infty} d(n)z^{-n/M}$$

$$=\sum_{n=-\infty}^{\infty} c(n)\delta_M(n)z^{-n/M}=\sum_{n=-\infty}^{\infty} c(n)\frac{1}{M}\sum_{k=0}^{M-1} e^{j2nk\pi/M}z^{-n/M}$$

$$=\frac{1}{M}\sum_{k=0}^{M-1}\sum_{n=-\infty}^{\infty} c(n)\left(e^{-j2k\pi/M}z^{1/M}\right)^{-n}=\frac{1}{M}\sum_{k=0}^{M-1} C\left(e^{-j2k\pi/M}z^{1/M}\right)$$

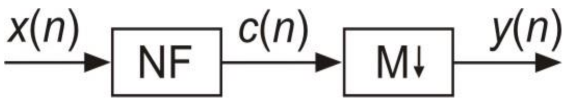
$$Y(\omega)=\frac{1}{M}\sum_{k=0}^{M-1} C\left(\frac{\omega-2k\pi}{M}\right)$$

$$C(\omega)=X(\omega)H_{NF}(\omega)\quad\Rightarrow\quad Y(\omega)=\frac{1}{M}\sum_{k=0}^{M-1} C(\omega/M)$$



Decimation

- Decreasing the sampling frequency *M* times in the digital domain



$$Y(z)=\sum_{n=-\infty}^{\infty} y(n)z^{-n}=\sum_{n=-\infty}^{\infty} c(Mn)z^{-n}=\sum_{n=-\infty}^{\infty} d(Mn)z^{-n}$$

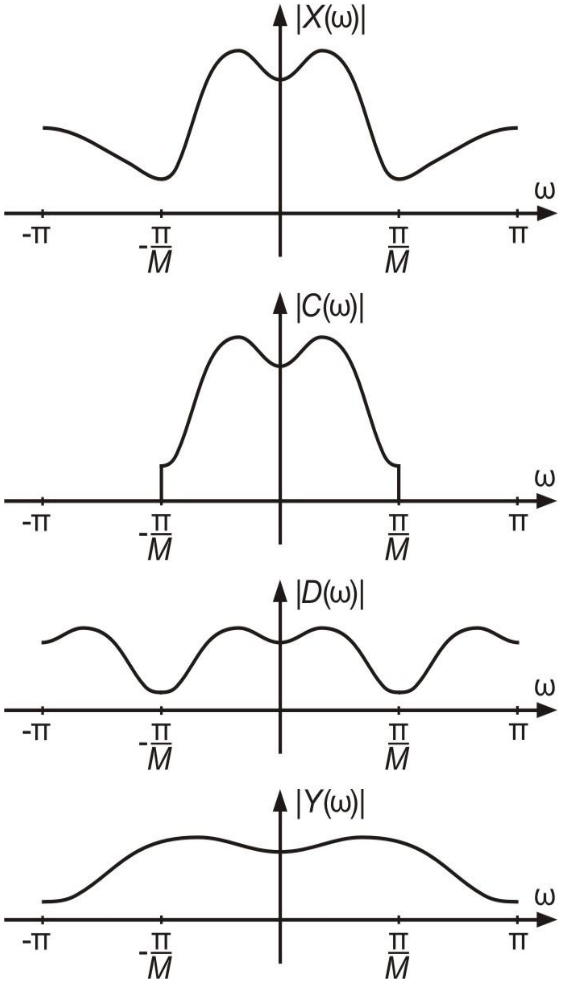
$$=\sum_{\substack{m=-\infty\\ m=kM}}^{\infty} d(m)z^{-m/M}=\sum_{n=-\infty}^{\infty} d(n)z^{-n/M}$$

$$=\sum_{n=-\infty}^{\infty} c(n)\delta_M(n)z^{-n/M}=\sum_{n=-\infty}^{\infty} c(n)\frac{1}{M}\sum_{k=0}^{M-1} e^{j2nk\pi/M}z^{-n/M}$$

$$=\frac{1}{M}\sum_{k=0}^{M-1}\sum_{n=-\infty}^{\infty} c(n)(e^{-j2k\pi/M}z^{1/M})^{-n}=\frac{1}{M}\sum_{k=0}^{M-1} C(e^{-j2k\pi/M}z^{1/M})$$

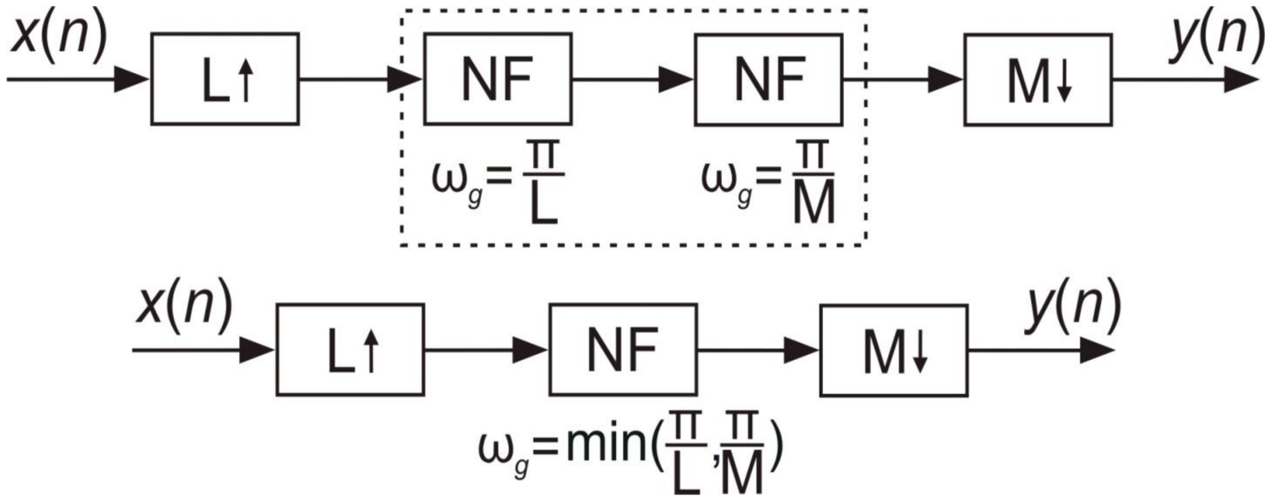
$$Y(\omega)=\frac{1}{M}\sum_{k=0}^{M-1} C\left(\frac{\omega-2k\pi}{M}\right)$$

$$C(\omega)=X(\omega)H_{NF}(\omega)\quad\Rightarrow\quad Y(\omega)=\frac{1}{M}\sum_{k=0}^{M-1} C(\omega/M)$$



General sampling rate conversion

- The ratio between the old and the new sampling rate is not always an integer
- If the ratio is a (convenient) rational number, the change in f_s can be effected by combining decimation and interpolation



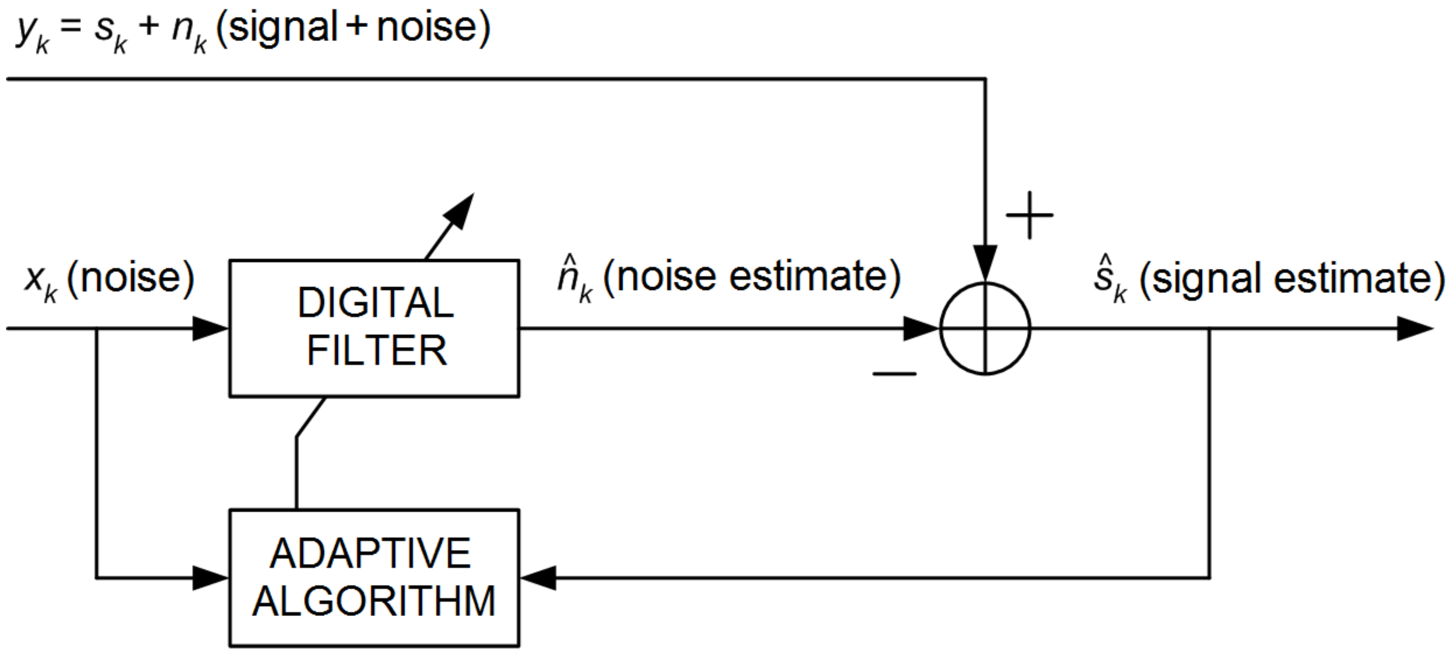
ADAPTIVE SYSTEMS

Definition and applications

- Discrete-time system whose characteristics automatically change depending on certain characteristics of the input signal
  - Filter with time-varying coefficients
  - Filter component is usually FIR
    - Simple and without risk of instability
- An adaptive system is not LTI
- Applications
  - When the problem demands the system to be adaptable to changing conditions
    - Equalization of a signal at the output of a channel of unknown properties
  - When it is necessary to reduce the noise whose spectrum overlaps with the spectrum of the useful signal, or its position changes with time
    - ECG, EEG
    - Communication in spread spectrum
    - Echo cancellation

Examples

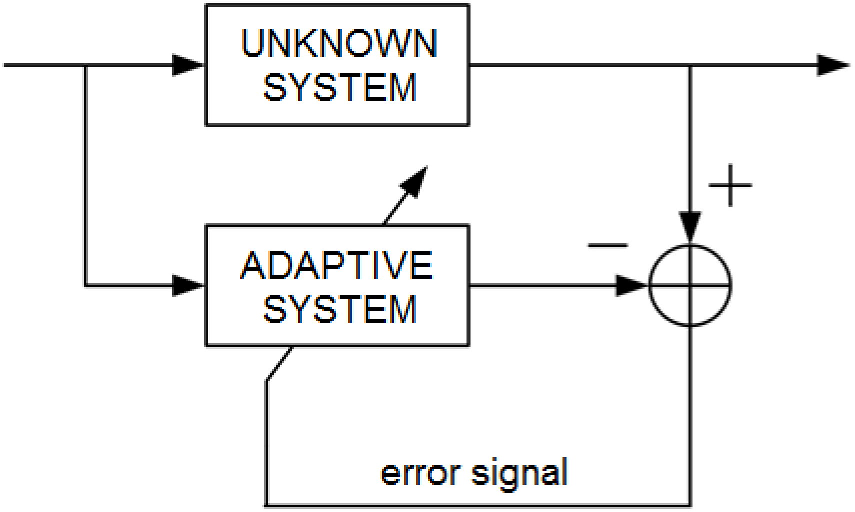
- Adaptive system for noise cancellation



  - x_k is the measure of noise, which is somehow correlated with n_k
  - The aim is to make an optimal estimate of noise in the input signal
  - $\hat{s}_k$ serves as the estimate of the output signal, and as the error signal as well

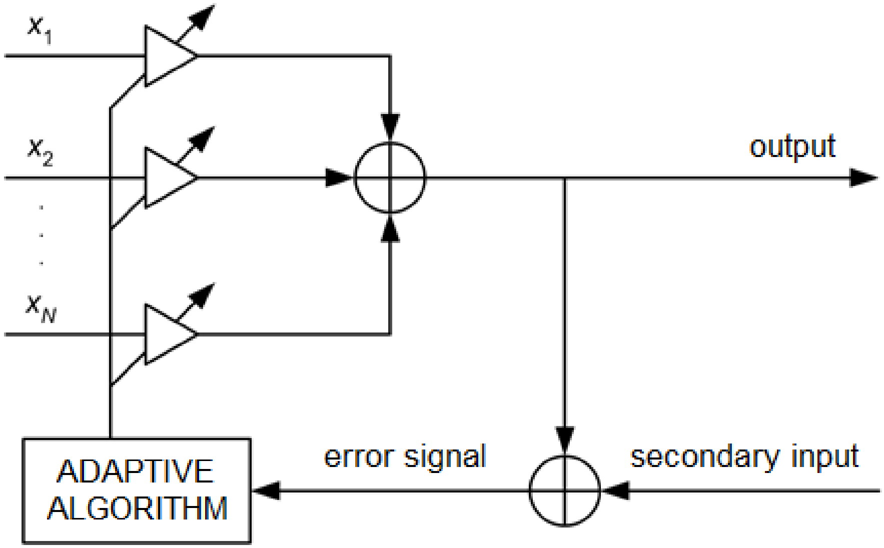
Examples

- System modelling



  - Estimating the transfer function of an unknown system

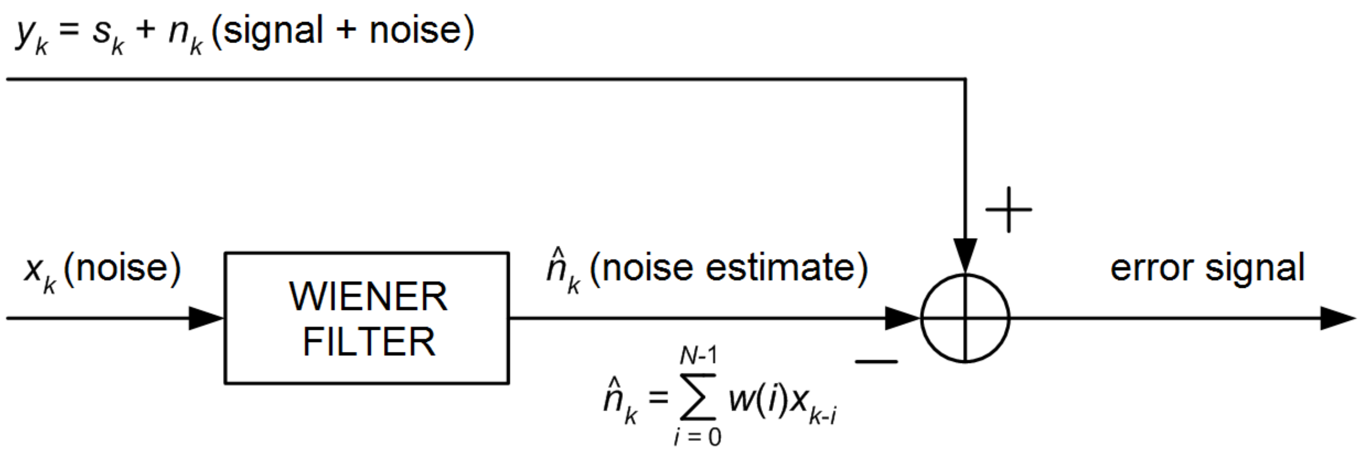
- Linear combiner



  - Adaptive modification of coefficients that multiply input signals

Adaptive algorithms

- All these algorithms aim to minimize the error signal according to a certain criterion
  - LMS (least mean square) algorithms
    - Most efficient
    - Do not have the problem of numeric instability
  - RLS (recursive least square) algorithms
  - Kalman filters
- Many of these can be seen as approximations of Wiener filter



FILTER DESIGN

Frequency response

- DTFT of the impulse response of an LTI system represents its *frequency response*

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n} = |H(\omega)| e^{j\Phi(\omega)}$$

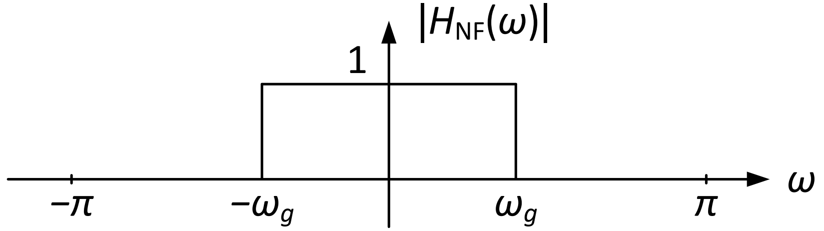
$|H(\omega)|$ – magnitude response

$\Phi(\omega)$ – phase response

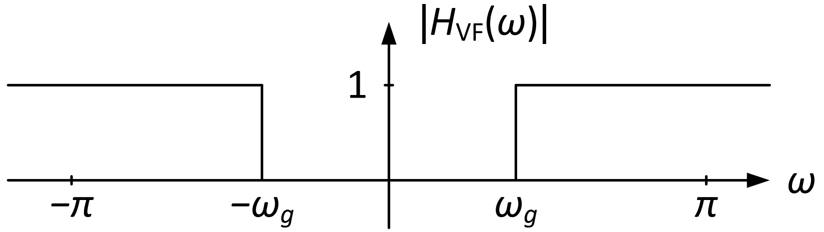
Ideal filters

- Although impossible to obtain in practice, they are often used as a starting point in filter design

$$H_{NF}(\omega) = \begin{cases} e^{-j\omega\tau}, & |\omega| \le \omega_g \\ 0, & \text{elsewhere} \end{cases}$$



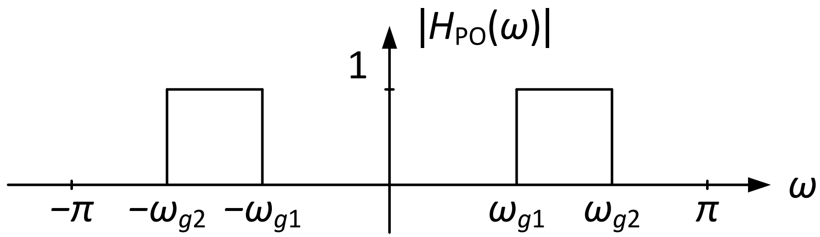
$$H_{VF}(\omega) = \begin{cases} e^{-j\omega\tau}, & |\omega| \ge \omega_g \\ 0, & \text{elsewhere} \end{cases}$$



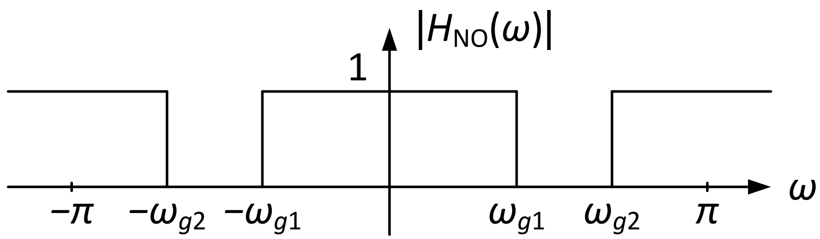
- Term $e^{-j\omega\tau}$ models the delay, which is defined to be constant in ideal filters

Ideal filters

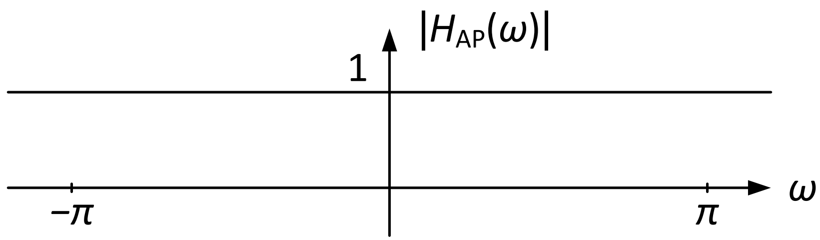
$$H_{\mathrm{PO}}(\omega)=\begin{cases} e^{-j\omega\tau}, & \omega_{g1}\le|\omega|\le\omega_{g2} \\ 0, & \text{elsewhere}\end{cases}$$



$$H_{\mathrm{NO}}(\omega)=\begin{cases} 0, & \omega_{g1}\le|\omega|\le\omega_{g2} \\ e^{-j\omega\tau}, & \text{elsewhere}\end{cases}$$



$$H_{\mathrm{AP}}(\omega)=e^{-j\omega\tau}$$



- In practice, we design filters starting from filter *specifications*, which define how much the frequency response can differ from the ideal one

Steps in filter design

- Design aim: to obtain an economical system that will meet the requirements
- Design stages:
  - Specification definition
    - Result: tolerance diagram
  - Approximation
    - Result: a stable rational transfer function
  - Choice of realization structure
    - Result: a block diagram with known values of multiplication constants
  - Quantization and verification
    - Result: answer to the question whether a filter with given finite wordlength will still meet the requirements defined in the first phase
  - Implementation (hardware or software)
    - Result: the realized system
- If needed, certain steps can be repeated

FIR or IIR filter?

- FIR are always stable
- FIR can achieve constant delay (linear phase response)
  - Impulse response should be either symmetrical or anti-symmetrical
- FIR are much less sensitive to finite wordlength effects

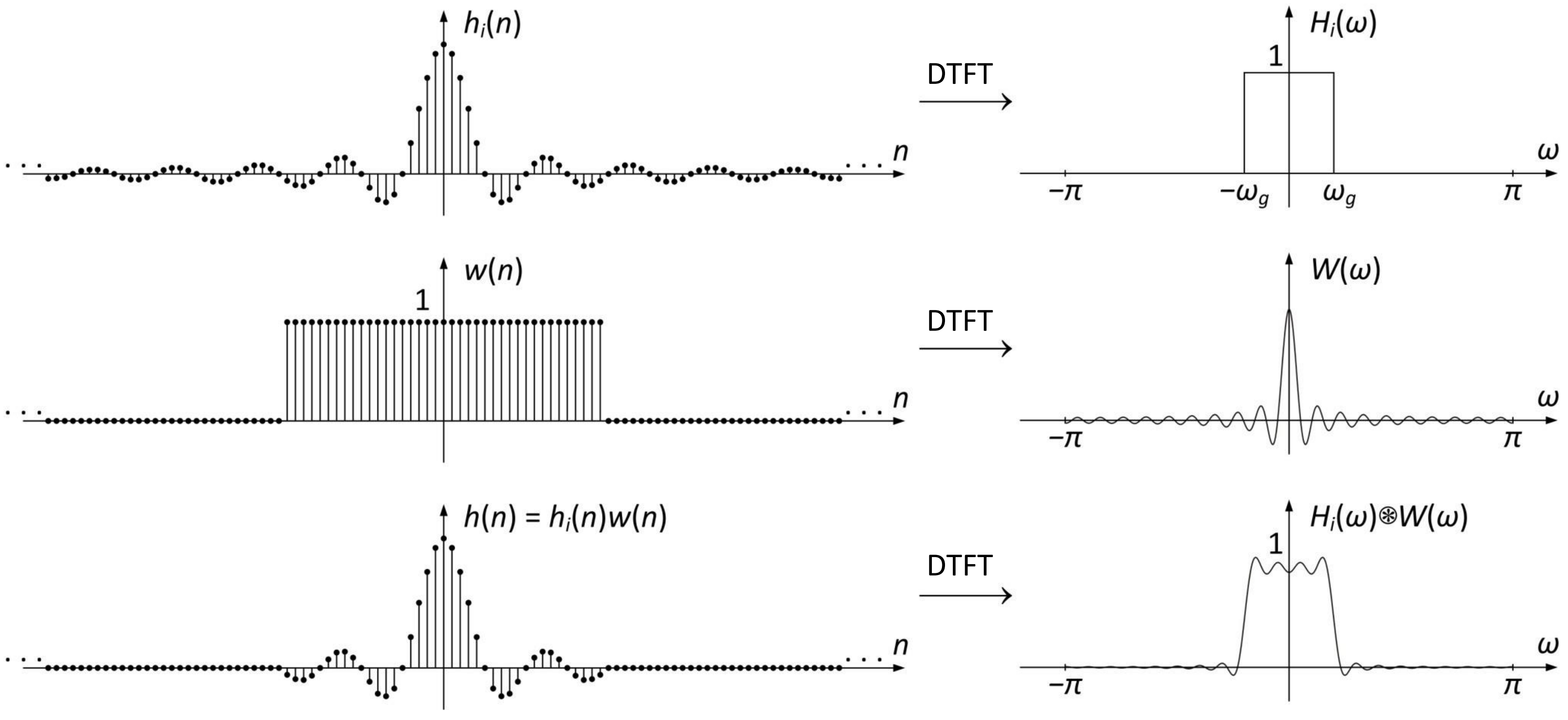
- IIR can generally meet a similar set of specifications with a lower order
  - IIR generally require less processing power and less work to set up

FIR filter design

- $H(\omega)$ can be specified in some points only
  - $h(n)$ can be found through a set of equations
  - If we have N equidistant points at the interval $0 \le \omega < 2\pi$, $h(n)$ can be obtained using inverse DFT of the sequence $\{H(\omega_k)\}$, this is called **sampling in frequency domain**
- $H(\omega)$ can be specified at the entire range $0 \le \omega < 2\pi$ with certain tolerance
  - $h(n)$ can be found using inverse DTFT of $\{H(\omega)\}$, and performing **windowing** on the obtained sequence of infinite duration
  - $h(n)$ can be found by iteratively solving a system of equations in order to meet a certain optimization criterion (**optimal method** of filter design)

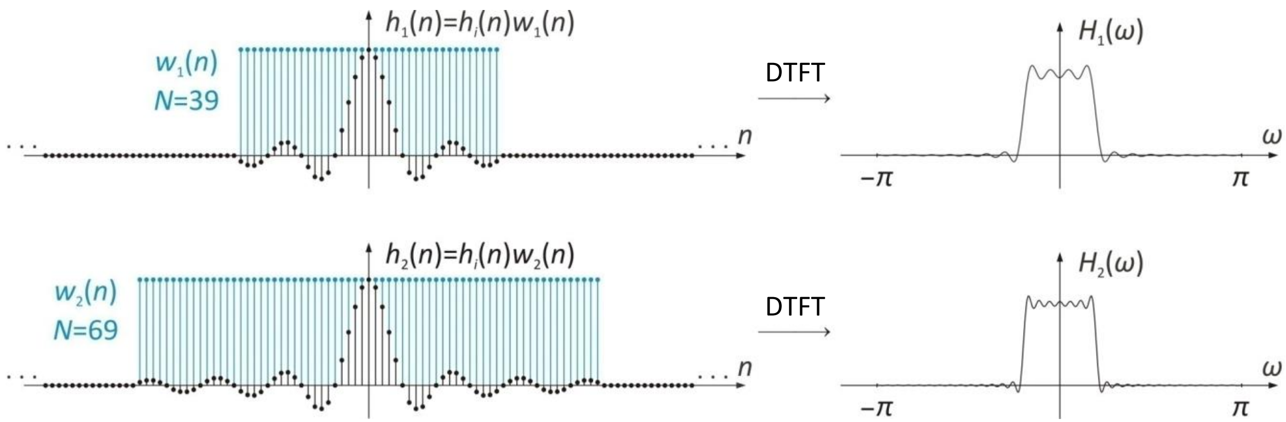
Example: FIR filter design by windowing

- Windowing causes distortion of the frequency response (spectral leakage and ripple)

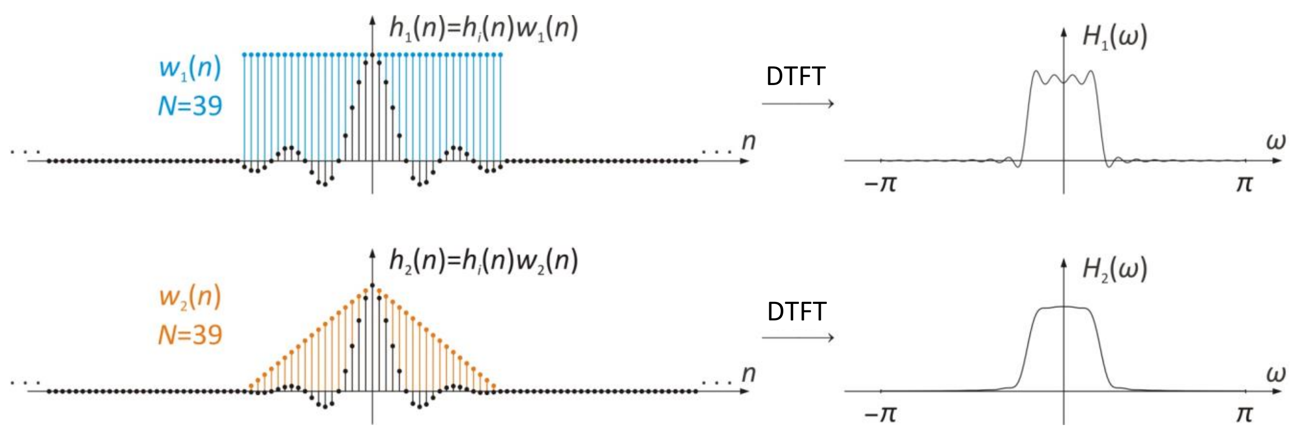


Example: FIR filter design by windowing

- Influence of window length



- Influence of window type



IIR filter design

- Methods for FIR filter design are not applicable here
- Predominant approach is to design an IIR filtar on the basis of a referent *continuous-time* prototype filter
  - There are recipes for standard frequency responses (low-pass, high-pass, band-pass, band-stop)
  - There are efficient algorithms for non-standard frequency responses as well
  - Direct design in polar coordinate system is avoided