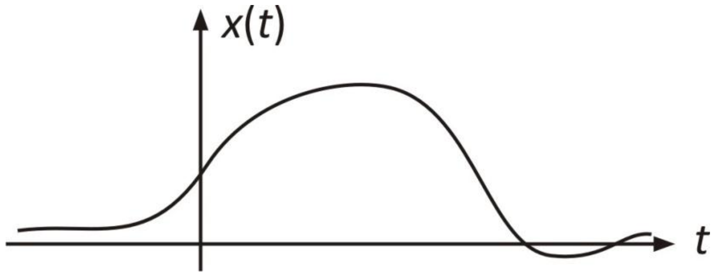
DISCRETE FOURIER TRANSFORM

Calculation of the spectrum in practice

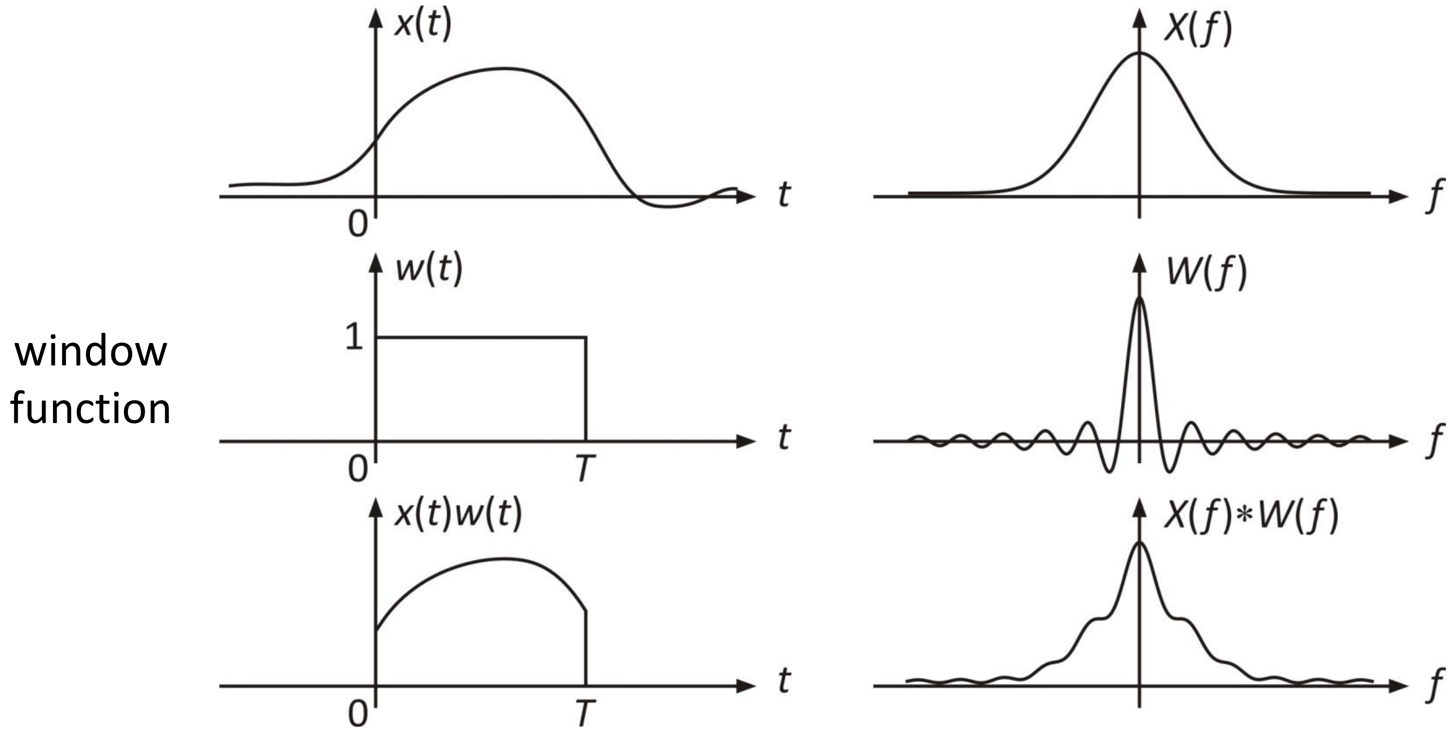


$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

- We should find a way to calculate (at least approximately) the Fourier transform of any signal on a computer (or a DSP)
- This implies the following:
  - **The spectrum should be calculated on the basis of the *samples* of the input signal**
  - **Only a *finite number* of samples should be used for calculation**
  - **The results should be obtained as a discrete *array of values***
  - The calculation should be performed *as efficiently as possible* (in terms of speed and memory)
  - The influence of the finite word length should be minimized

Calculation of the spectrum in practice

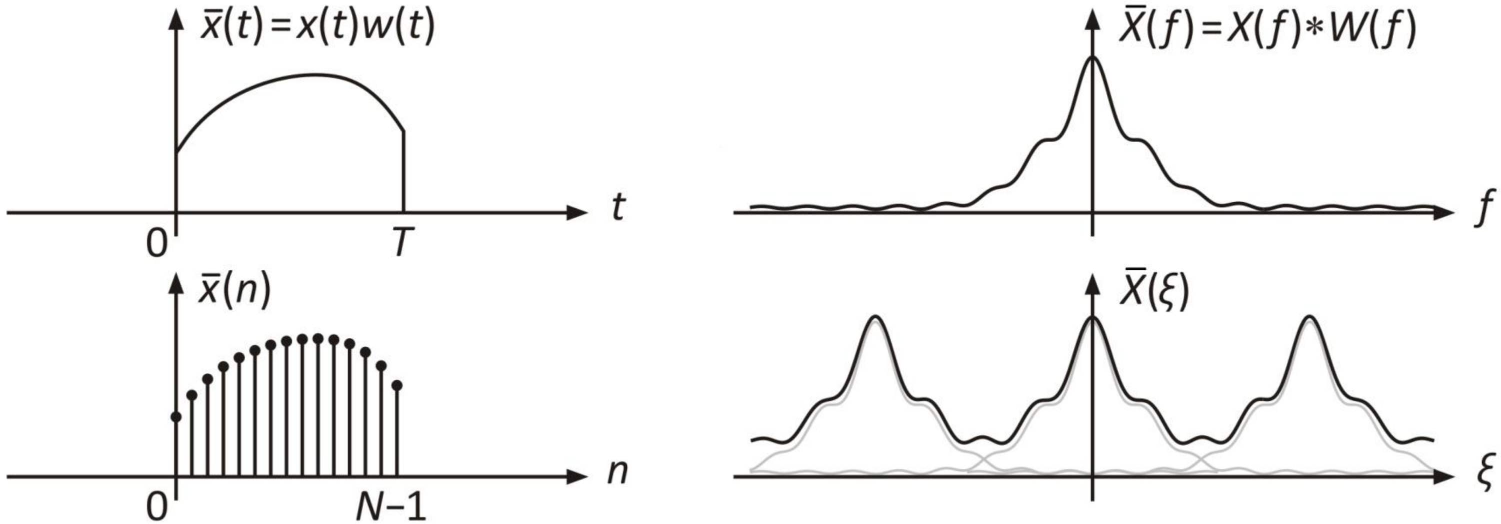
Limiting the duration of the signal (windowing)



$$X(f)*W(f)=\int_{-\infty}^{\infty}x(t)w(t)e^{-j2\pi ft}dt=\int_{0}^{T}x(t)e^{-j2\pi ft}dt\approx X(f)$$

Calculation of the spectrum in practice

Sampling of the signal



- Calculation of the spectrum on the basis of signal samples is equivalent to using DTFT instead of classical FT

$$\bar{X}(f) = \int_{-\infty}^{\infty} \bar{x}(t) e^{-j2\pi f t} dt \quad \longrightarrow \quad \bar{X}(\xi) = \sum_{n=-\infty}^{\infty} \bar{x}(n) e^{-j2\pi \xi n}$$

Calculation of the spectrum in practice

Sampling of the signal

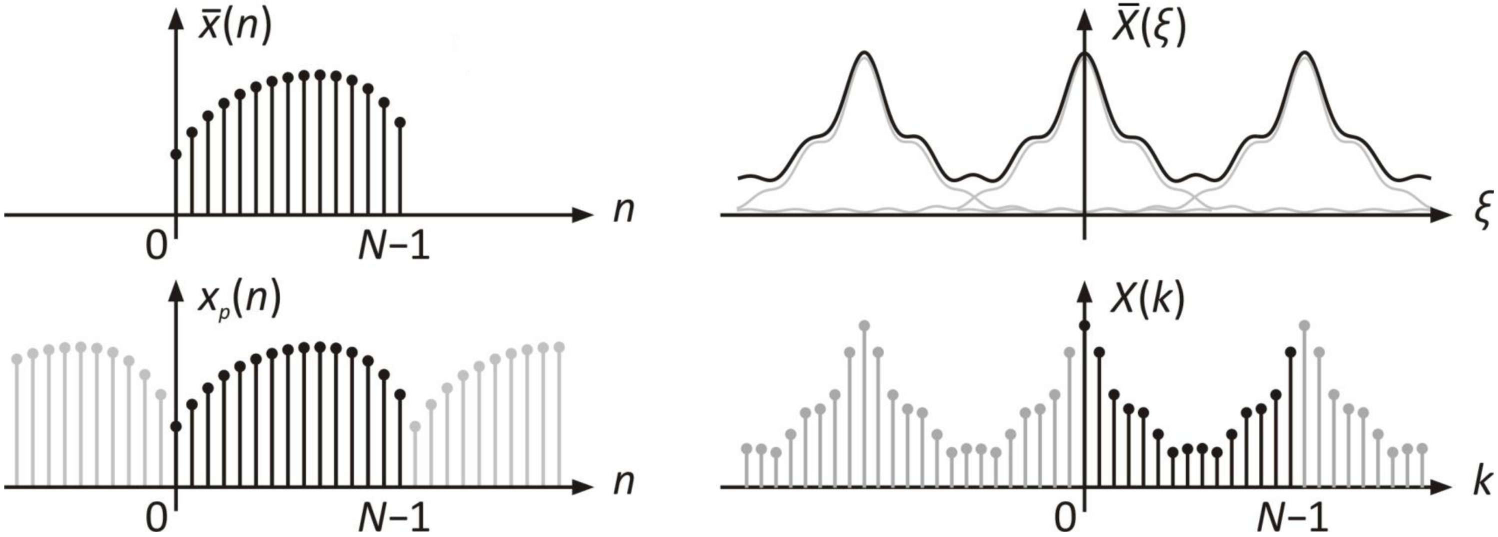
- The expression for DTFT is obtained from the expression for the spectrum of a sampled continuous-time signal:

$$\hat{X}(f) = \sum_{n=-\infty}^{\infty} x(nT) e^{-j2\pi f n T} = \frac{1}{T} \sum_{m=-\infty}^{\infty} X(f - m f_s)$$

- If the condition of the sampling theorem is not met, *aliasing* occurs (copies of the original spectrum overlap)
- The high-frequency region of the spectrum is usually the one more affected by aliasing
- It becomes impossible to extract the original spectrum even by using an ideal low-pass filter

Calculation of the spectrum in practice

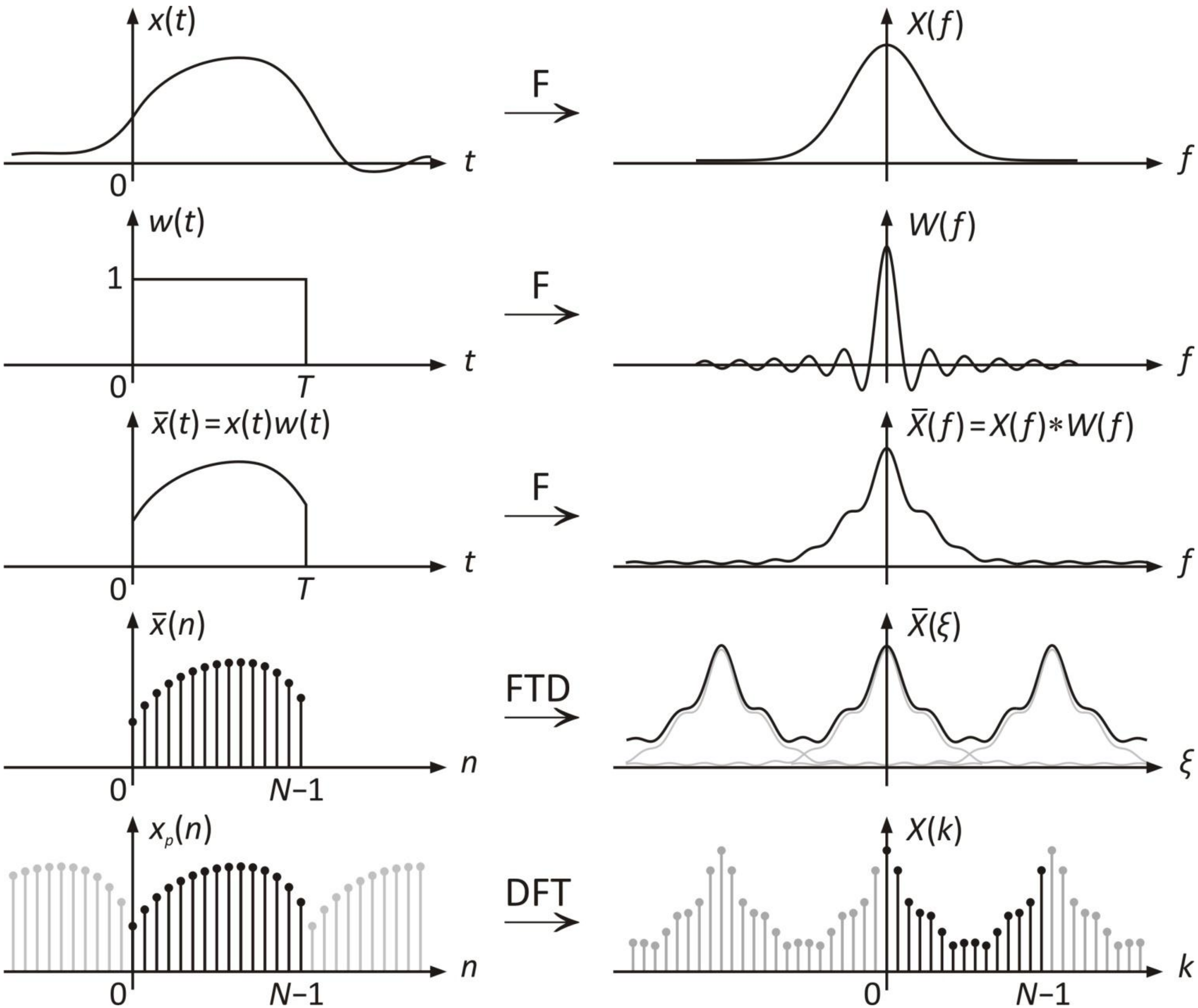
Spectrum discretization



- Spectrum is calculated only in a finite number of points $\xi_k = k/N$, $k = 0, 1, \ldots N-1$.

$$X(k) = \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi \frac{k}{N} n} \qquad x_p(n) = \sum_{m=-\infty}^{\infty} \bar{x}(n - mN)$$

Calculation of the spectrum in practice



Influence of windowing

$$\bar{x}(n) = x(n)w(n)$$

$$\bar{X}(\xi) = \sum_{n=-\infty}^{\infty} x(n)w(n)e^{-j2\pi\xi n} = \sum_{n=0}^{N-1} x(n)w(n)e^{-j2\pi\xi n} \approx X(\xi)$$

$$\bar{X}(\xi) = X(\xi) \circledast W(\xi) = \int_{-1/2}^{1/2} X(\lambda)W(\xi - \lambda)d\lambda$$

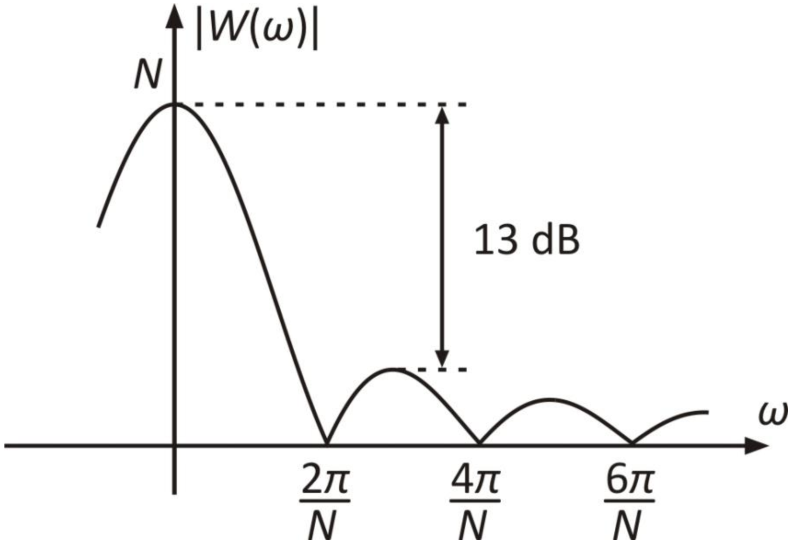
- Convolution distorts the original spectrum
- This influence will be analyzed for the discrete-time case, which does not make a big difference

Rectangular window function

$$w(n)=\begin{cases}1, & 0\le n\le N-1\\ 0, & \text{elsewhere}\end{cases}$$

$$W(z)=\sum_{n=0}^{N-1} z^{-n}=\frac{1-z^{-N}}{1-z^{-1}}$$

$$W(\omega)=\frac{1-e^{-jN\omega}}{1-e^{-j\omega}}=\frac{\sin\frac{N\omega}{2}}{\sin\frac{\omega}{2}}e^{-j\frac{N-1}{2}\omega}$$



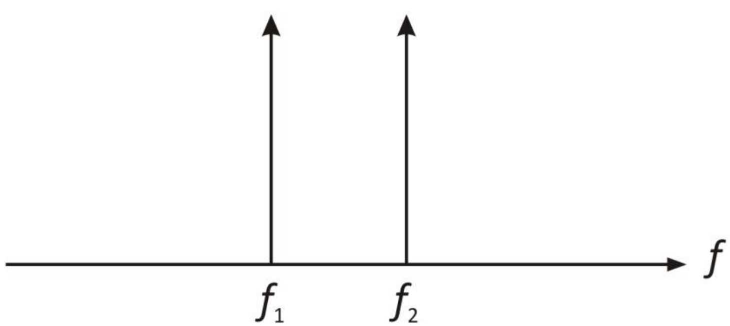
Zeros of $W(\omega)$ are located at frequencies $\omega=\frac{2\pi}{N}k,\quad k=\pm1,\pm2,...$

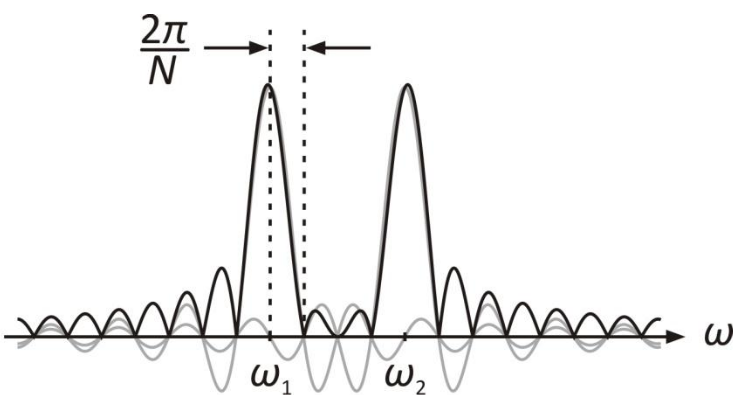
$$\left|\frac{W(\omega)}{W(0)}\right|_{\omega=\frac{3\pi}{N}}=\left|\frac{\sin 3\pi/2}{\sin 3\pi/2N}\right|:N\approx\frac{1}{N\cdot 3\pi/2N}=\frac{2}{3\pi}\approx 0{,}21 \qquad 20\log(0{,}21)\approx -13\,\mathrm{dB}$$

Frequency resolution

- The width of the main arcade of the window function defines the *frequency resolution*
- The minimum distance between two components which will still be distinguishable in the spectrum is:

$$\Delta\omega \geq \frac{2\pi}{N} \qquad \Delta f \geq \frac{f_s}{N} = \frac{1}{NT} = f_2 - f_1 \text{ [Hz]}$$





Frequency resolution

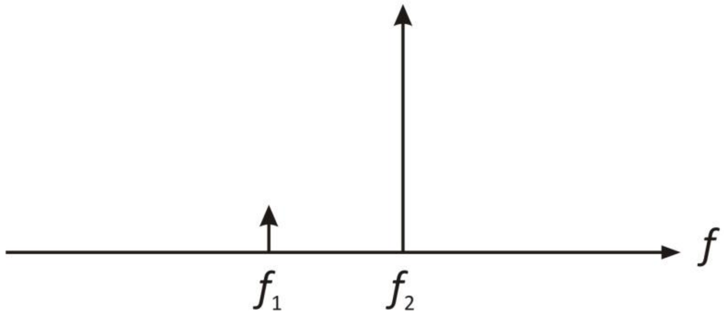
- Minimum length of the rectangular window function for the desired frequency resolution Δf ($\Delta\omega$) is:

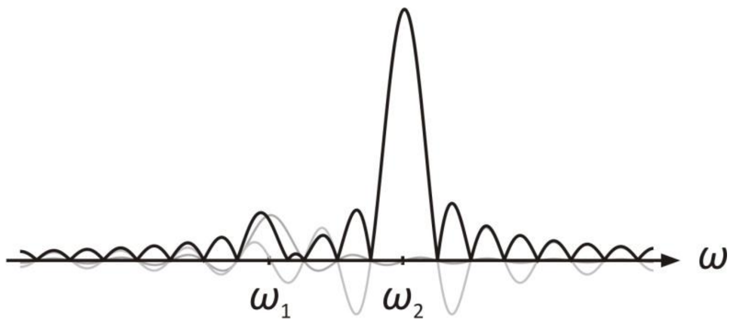
$$N \geq \frac{f_s}{\Delta f} = \frac{2\pi}{\Delta\omega}$$

- How to improve frequency resolution (reduce $\Delta f = f_s / N$)?
  - Decrease f_s (increases the effects of sampling)
  - Increase N (requires more calculations, and sometimes it is not even possible due to limited availability of samples)
- Improved frequency resolution decreases temporal resolution

Spectral leakage (DTFT)

- Consequence of the existence of side lobes in the spectrum of the window function
- The spectrum of the windows signal may contain components even at frequencies at which the original spectrum was zero
  - Main arcades of weaker spectral component may be completely hidden





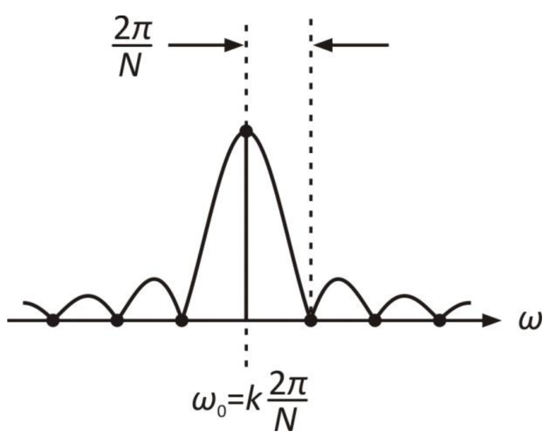
- The effect may be reduced by using better window functions
  - There are window functions with sidelobes suppressed by nearly 100 dB

Spectral leakage (DFT)

- It remains to be seen whether the DFT samples of the DTFT spectrum will faithfully represent the amplitudes of peaks in the DTFT spectrum
  - Convolution of the spectrum of the window function with a component in the original spectrum should appear at the location of a DFT sample
  - DFT samples are located at frequencies $\omega_k = 2k\pi/N$
  - The components in the original spectrum should also appear at $\omega_k = 2k\pi/N$ in order to be faithfully represented by the DFT spectrum

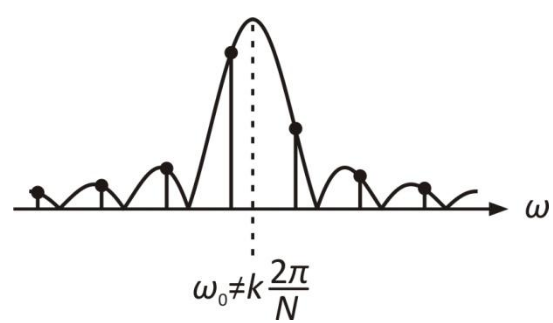
$$f_0 = k\frac{f_s}{N}$$

$$N = kN_0$$



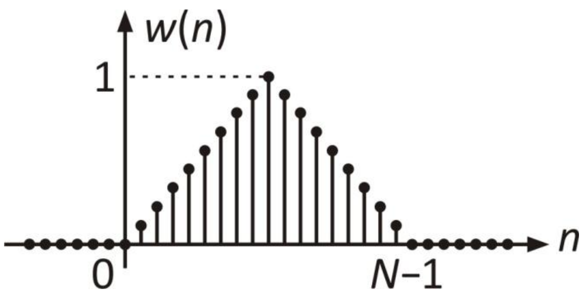
$$f_0 \neq k\frac{f_s}{N}$$

$$N \neq kN_0$$



Bartlett (triangular) prozorska funkcija

$$w(n) = \begin{cases} \dfrac{2n}{N-1}, & 0 \le n \le \dfrac{N-1}{2} \\ 2 - \dfrac{2n}{N-1}, & \dfrac{N-1}{2} < n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$$



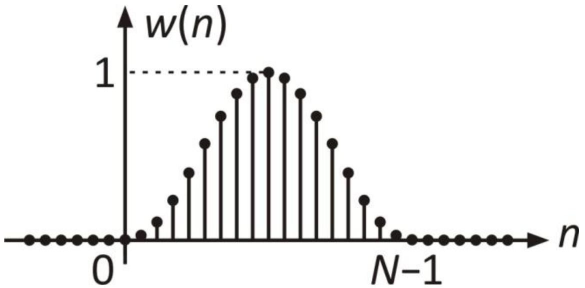
- Side lobes are attenuated by 26 dB
- Main lobe is approximately twice as wide, which deteriorates frequency resolution approximately twice
  - This can be compensated by taking twice as many samples

$$\Delta\omega \ge \frac{4\pi}{N} \qquad \Delta f \ge 2\frac{f_s}{N} \qquad N \ge 2\frac{f_s}{\Delta f}$$

Some other window functions

Hann window function

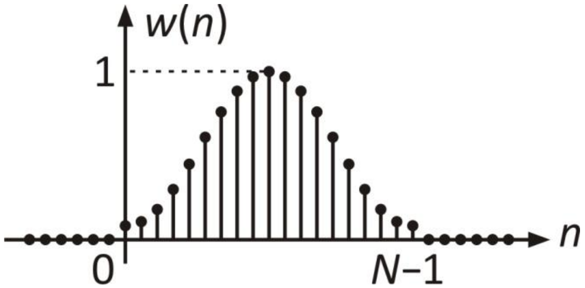
$$w(n)=\begin{cases}0{,}5-0{,}5\cos\dfrac{2\pi n}{N-1}, & 0\le n\le N-1\\ 0, & \text{elsewhere}\end{cases}$$



- Side lobe attenuation: 31 dB
- Frequency resolution as with Bartlett window

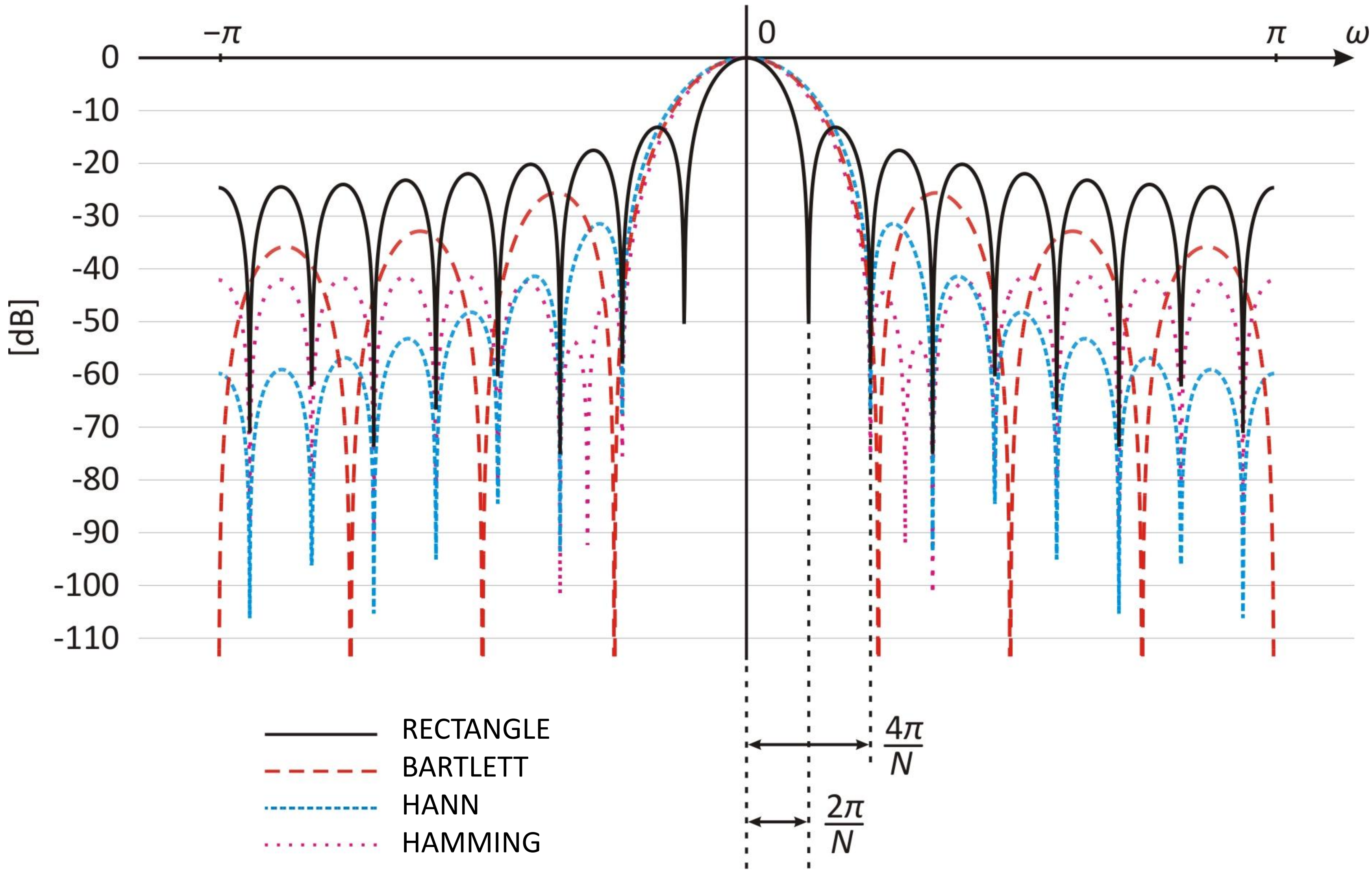
Hamming window function

$$w(n)=\begin{cases}0{,}54-0{,}46\cos\dfrac{2\pi n}{N-1}, & 0\le n\le N-1\\ 0, & \text{elsewhere}\end{cases}$$



- Side lobe attenuation: 43 dB
- Frequency resolution as with Bartlett window

Spectra of window functions



Frequency resolution (example 1)

- The signal containing 4 sinusoids at frequencies 1, 1.5, 2.5 and 2.75 kHz, is sampled with f_s = 10 kHz. What is the minimum number of samples of this signal to be taken for the 4 spectral peaks to remain visible:
  a) if rectangular window is used;
  b) if Hamming window is used?

a) $$N \geq \frac{f_s}{\Delta f} = \frac{10}{0.25} = 40 \text{ samples}$$

b) $$N \geq 2\frac{f_s}{\Delta f} = \frac{20}{0.25} = 80 \text{ samples}$$

Frequency resolution (example 2)

- A 10-ms segment of a continuous-time signal is sampled with f_s = 10 kHz. The signal contains sinusoids at f_1 = 1 kHz and f_2 = 2 kHz as well as at a frequency f_3 which lies between f_1 and f_2. How close can f_3 get to f_1 or f_2, for individual spectral peaks to remain visible, if a rectangular window is used?

$$N = f_s \cdot \Delta t = 10\,\text{kHz} \cdot 10\,\text{ms} = 100\,\text{samples}$$

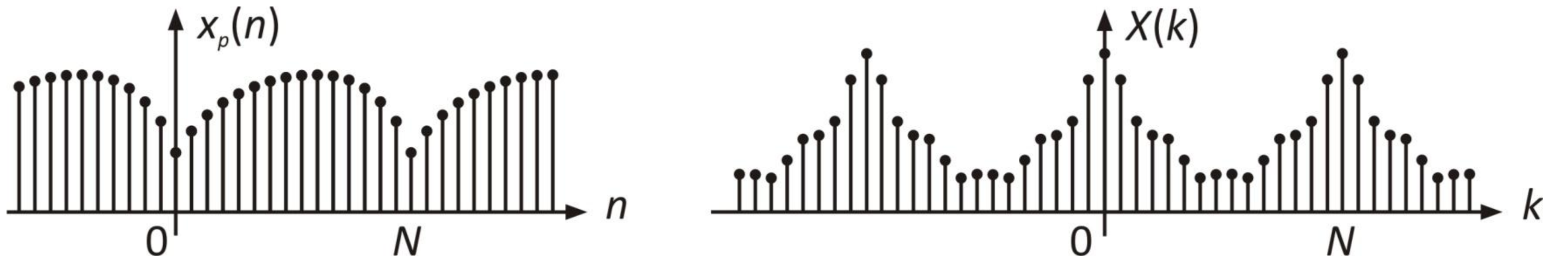
$$\Delta f = \frac{f_s}{N} = 100\,\text{Hz}$$

$$f_1 + \Delta f = 1.1\,\text{kHz} \le f_3 \le 1.9\,\text{kHz} = f_2 - \Delta f$$

How to obtain a discrete spectrum?

- Discrete nature of a spectrum generally comes from periodicity in the signal
  - Periodic continous-time signals can have infinitely many harmonics in the spectrum
  - Periodic discrete-time signals can have only a finite number of them
    - *k*-th component in the spectrum corresponds to the frequency $2k\pi/N$
    - After N components they begin repeating

$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}nk}$$



Fourier expansion of a periodic signal

$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}nk}$$

What are the values of coefficients $X(k)$?

$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-nk} \qquad W_N = e^{-j\frac{2\pi}{N}}$$

$$\sum_{n=0}^{N-1} x_p(n) W_N^{nr} = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} X(k) W_N^{-(k-r)n}$$

$$= \sum_{k=0}^{N-1} X(k)\left[\frac{1}{N}\sum_{n=0}^{N-1} W_N^{-(k-r)n}\right] = X(r)$$

$$X(k) = \sum_{n=0}^{N-1} x_p(n) W_N^{nk}$$

Discrete Fourier transform (DFT)

- Mapping of a periodic discrete signal $x(n)$ of period N into a sequence of complex numbers:

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}nk}$$

- The obtained sequence is also periodic with period N
- Inverse DFT is given by:

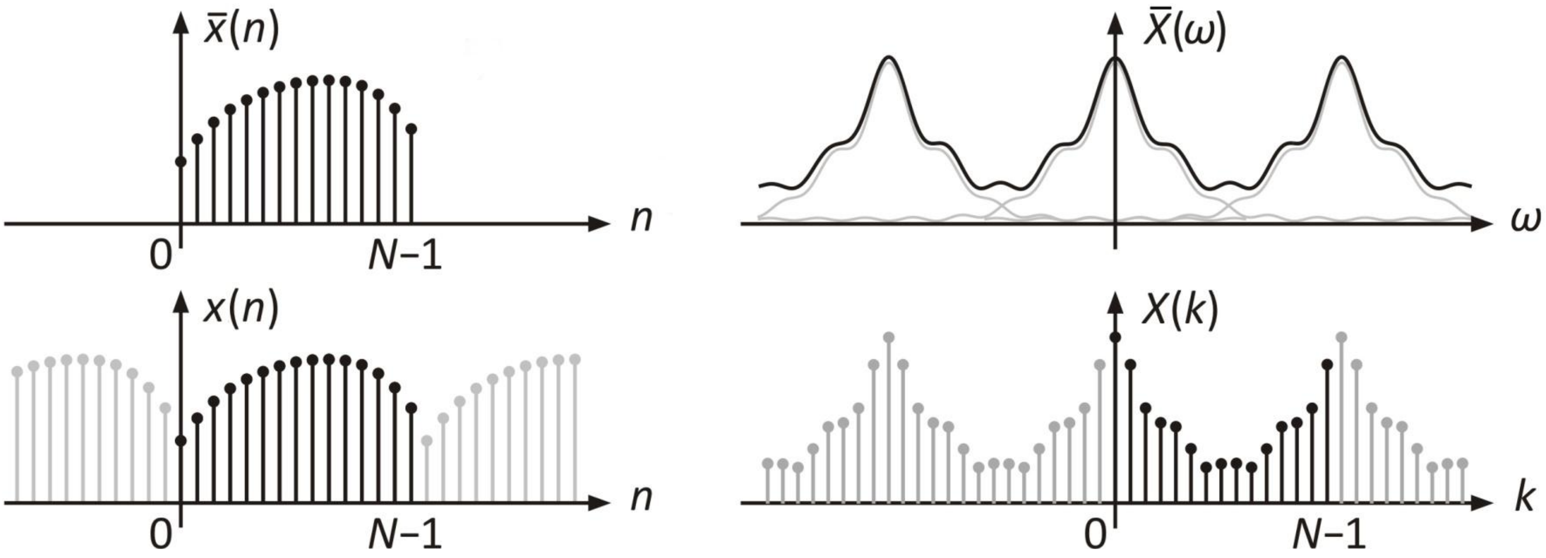
$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}nk}$$

Relationship between DFT and FTD

- DFT actually represents a discrete version of DTFT, since:

$$X(k) = \bar{X}(\omega)\Big|_{\omega = k\frac{2\pi}{N}}$$

where $\bar{x}(n)$ is the signal which contains only the initial period of the signal $x(n)$



Properties of DFT

Periodicity

$$X(k) = X(k+N)$$

Linearity

$$\mathrm{DFT}\{ax(n)+by(n)\} = aX(k)+bY(k)$$

Time shifting

$$\mathrm{DFT}\{x(n-m)\} = e^{-j\frac{2\pi}{N}km}X(k)$$

Modulation

$$\mathrm{DFT}\{e^{j\frac{2\pi}{N}nl}x(n)\} = X(k-l)$$

Properties of DFT

DFT of a real signal

If the discrete-time signal $x(n)$ is real,

$$X(N-k) = X(-k) = X^*(k)$$ (Hermitian symmetry)

This amounts to: $\mathrm{Re}\{X(-k)\} = \mathrm{Re}\{X(k)\}$

$$\mathrm{Im}\{X(-k)\} = -\mathrm{Im}\{X(k)\}$$

which is equivalent to:

$$|X(-k)| = |X(k)|$$

$$\arg\{X(-k)\} = -\arg\{X(k)\}$$

Properties of DFT

Transform of convolution

$$\text{DFT}\{x(n) \circledast y(n)\} = X(k)Y(k)$$

Transform of product

$$\text{DFT}\{x(n)y(n)\} = \frac{1}{N} X(k) \circledast Y(k)$$

Parseval's theorem

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$