FOURIER TRANSFORM
OF DISCRETE-TIME SIGNALS

Fourier transform of continuous-time signals

$$X(\Omega) = \mathrm{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

$$x(t) = \mathrm{F}^{-1}\{X(\Omega)\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\Omega) e^{j\Omega t} d\Omega$$

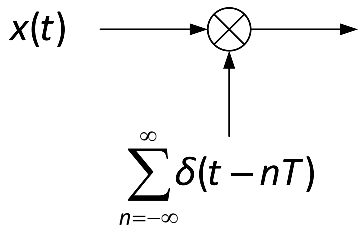
JEAN-BAPTISTE JOSEPH FOURIER
(1768-1830)

- A special case of Laplace transform

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \qquad X(\Omega) = X(s)\Big|_{s=j\Omega}$$

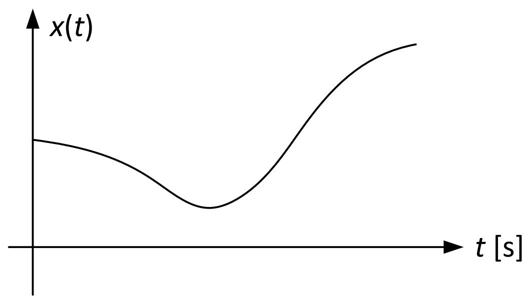
- Fourier transform of the impulse response of an LTI system represents its *frequency response*

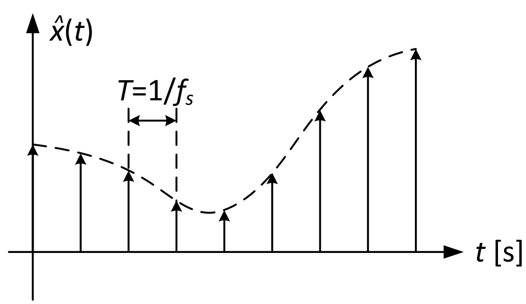
Spectrum of a sampled signal



$$\hat{x}(t)=x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

$$=\sum_{n=-\infty}^{\infty}x(nT)\delta(t-nT)$$





$$\hat{X}(f)=\mathrm{F}\{\hat{x}(t)\}=\int_{-\infty}^{\infty}\hat{x}(t)e^{-j2\pi ft}dt$$

$$\hat{X}(f)=\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}x(nT)\delta(t-nT)e^{-j2\pi ft}dt$$

$$\hat{X}(f)=\sum_{n=-\infty}^{\infty}x(nT)\int_{-\infty}^{\infty}\delta(t-nT)e^{-j2\pi ft}dt$$

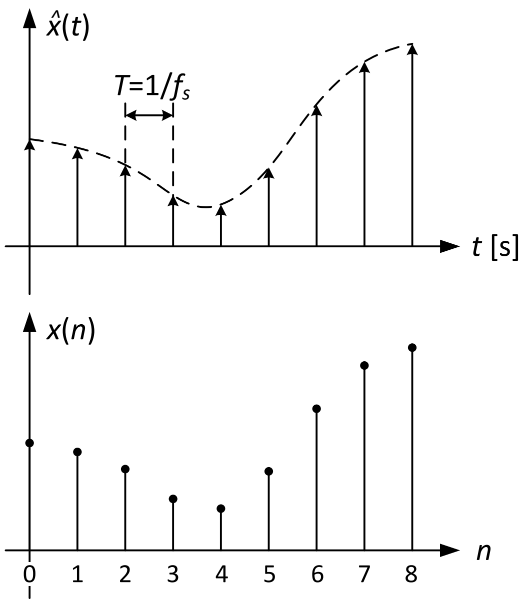
$$\hat{X}(f)=\sum_{n=-\infty}^{\infty}x(nT)e^{-j2\pi fnT}\overbrace{\int_{-\infty}^{\infty}\delta(t-nT)dt}^{1}$$

$$\boxed{\hat{X}(f)=\sum_{n=-\infty}^{\infty}x(nT)e^{-j2\pi fnT}}$$

FOURIER TRANSFORM OF A
DISCRETE-TIME SIGNAL

Discrete-time Fourier transform (DTFT)

- Based on the already obtained expression for the spectrum of a sampled continuous-time signal



$$\hat{X}(f) = \sum_{n=-\infty}^{\infty} x(nT) e^{-j2\pi f n T} \qquad \hat{X}(\Omega) = \sum_{n=-\infty}^{\infty} x(nT) e^{-jn\Omega T}$$

$$X(\xi) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi \xi n} \qquad \boxed{X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\omega}}$$

Discrete-time Fourier transform (DTFT)

- The expression converges if $|X(\omega)| < \infty$ for each *ω*. Having in mind the following:

$$|X(\omega)| = \left|\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}\right| \le \sum_{n=-\infty}^{\infty} |x(n)|\left|\cancelto{1}{e^{-j\omega n}}\right| = \sum_{n=-\infty}^{\infty} |x(n)|,$$

a sufficient condition for convergence is $\boxed{\sum_{n=-\infty}^{\infty} |x(n)| < \infty}$

- DTFT is *periodical* with the period of $\omega = 2\pi$ ($\xi = 1$)

$$X(\omega + 2k\pi) = \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega+2k\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}\cancelto{1}{e^{-j2k\pi n}} = X(\omega)$$

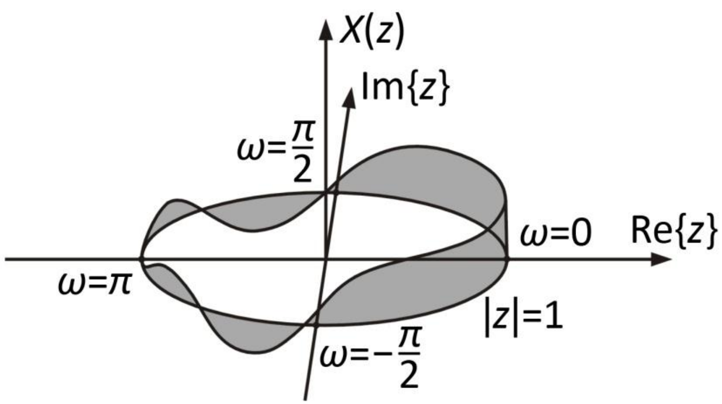
Relationship between z-transform and DTFT

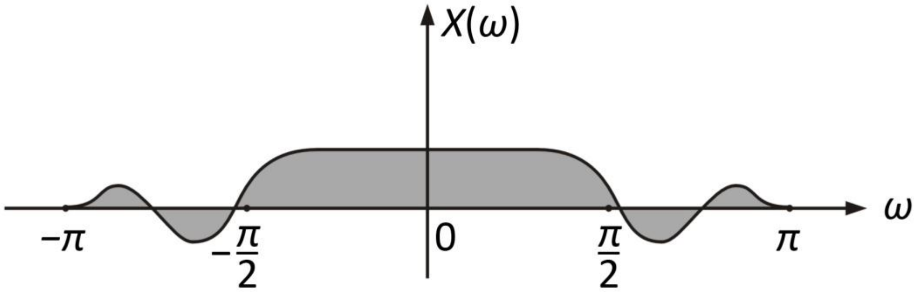
ZT $\quad X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

DTFT $\quad X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\omega}$

$$X(\omega) = X(z)\Big|_{z=e^{j\omega}}$$

- DTFT consists of the values of z-transform at the unit circle in the z-plain





Properties of DTFT

Linearity

$$\text{DTFT}\{ax(n)+by(n)\}=aX(\omega)+bY(\omega)$$

Time shifting

$$\text{DTFT}\{x(n-m)\}=e^{-j\omega m}X(\omega)$$

Modulation

$$\text{DTFT}\{e^{j\omega_0 n}x(n)\}=X(\omega-\omega_0)$$

Properties of DTFT

DTFT of a real signal

If the discrete/time signal $x(n)$ is real,

$$\boxed{X(-\omega) = X^*(\omega)}$$ (Hermitian symmetry)

This amounts to: $\mathrm{Re}\{X(-\omega)\} = \mathrm{Re}\{X(\omega)\}$

$$\mathrm{Im}\{X(-\omega)\} = -\mathrm{Im}\{X(\omega)\}$$

which is equivalent to:

$$|X(-\omega)| = |X(\omega)|$$

$$\arg\{X(-\omega)\} = -\arg\{X(\omega)\}$$

Properties of DTFT

Time reversal

$$\mathrm{DTFT}\{x(-n)\} = X(-\omega)$$

If the signal $x(n)$ is real, this amounts to:

$$\mathrm{DTFT}\{x(-n)\} = X^*(\omega)$$

Differentiation

$$\mathrm{DTFT}\{nx(n)\} = j\,\frac{dX(\omega)}{d\omega}$$

Properties of DTFT

Transform of convolution

$$\text{DTFT}\left\{x(n)*y(n)\right\}=X(\omega)Y(\omega)$$

Transform of product

$$\text{DTFT}\{x(n)y(n)\}=X(\omega)\circledast Y(\omega)$$

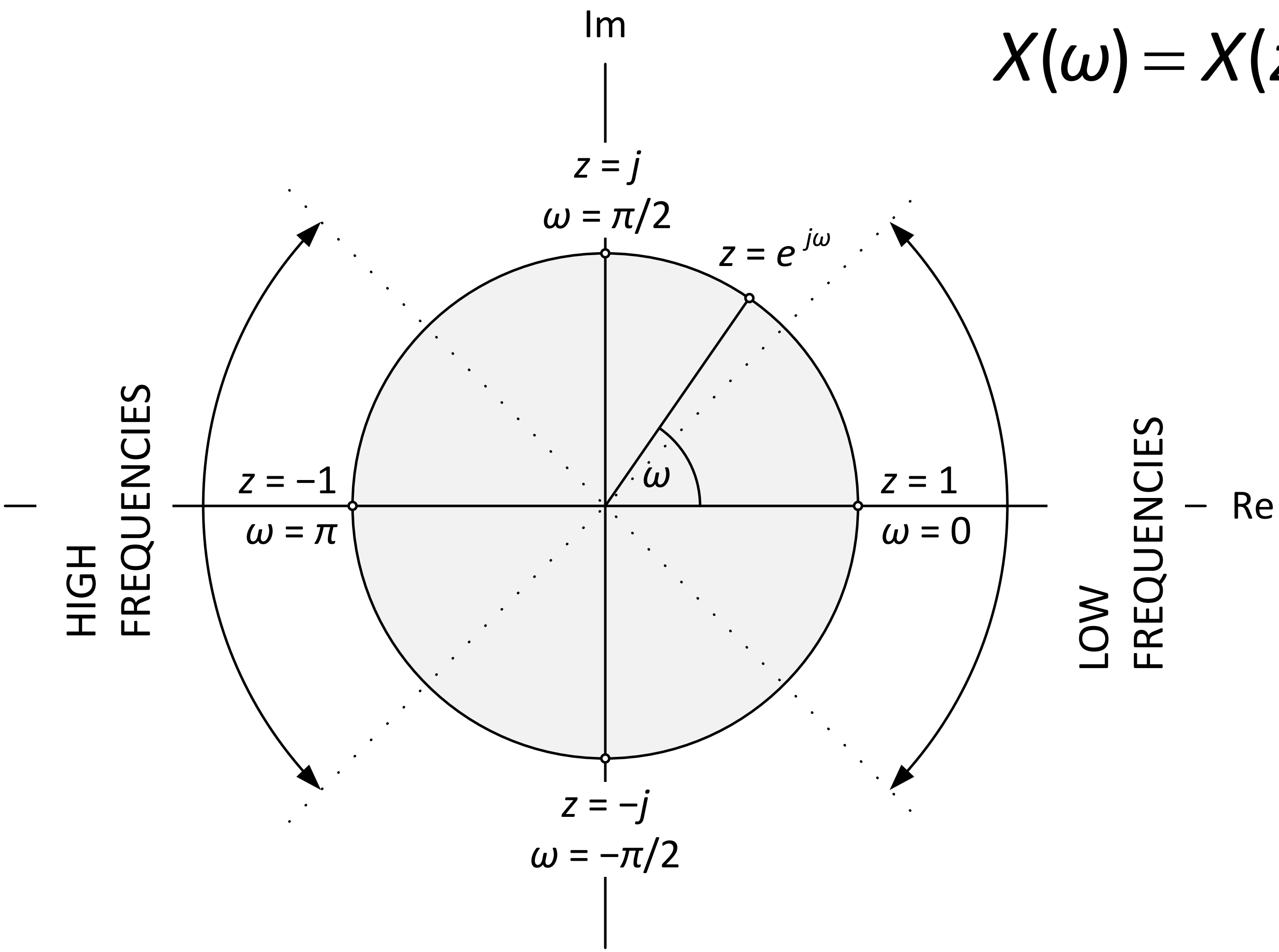
$$X(\omega)\circledast Y(\omega)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(\lambda)Y(\omega-\lambda)d\lambda$$

Parseval's theorem

$$\sum_{n=-\infty}^{\infty}\left|x(n)\right|^{2}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X(\omega)\right|^{2}d\omega$$

Relationship between z-transform and DTFT

$$X(\omega) = X(z)\Big|_{z=e^{j\omega}}$$



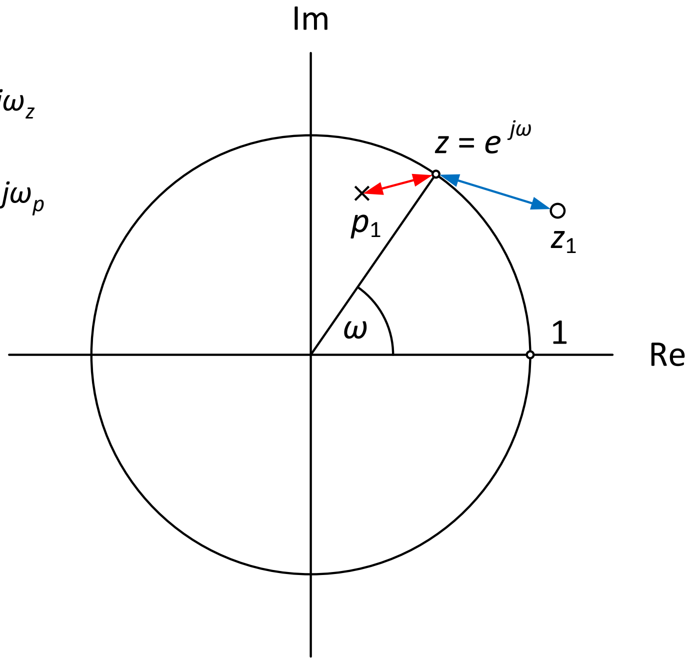
Influence of zeros and poles of ZT on DTFT

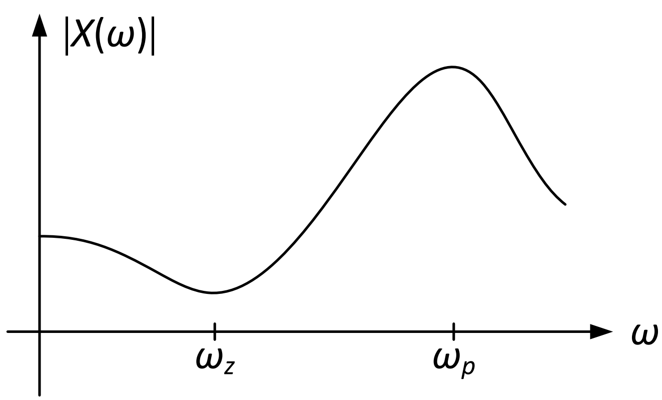
- Assuming a single pole and a single zero:

$$X(z)=\frac{z-z_1}{z-p_1} \qquad X(\omega)=\frac{e^{j\omega}-z_1}{e^{j\omega}-p_1} \Rightarrow \left|X(\omega)\right|=\frac{\left|e^{j\omega}-z_1\right|}{\left|e^{j\omega}-p_1\right|}$$

$$z_1=r_z e^{j\omega_z}$$

$$p_1=r_p e^{j\omega_p}$$





Influence of zeros and poles of ZT on DTFT

- Zeros *decrease* the value of the module of DTFT near the frequency where they are located, and this influence is stronger if they are nearer to the unit circle
- Poles *increase* the value of the module of DTFT near the frequency where they are located, and this influence is stronger if they are nearer to the unit circle

- This is all quite logical having in mind that:
  - The value of *z*-transform at its zero is equal to 0
  - The value of *z*-transform near its pole goes to infinity
  - DTFT is equal to *z*-transform at the unit circle

Frequency response

- DTFT of the impulse response of an LTI system represents its *frequency response*

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n} = |H(\omega)| e^{j\Phi(\omega)}$$

$|H(\omega)|$ – magnitude response

$\Phi(\omega)$ – phase response

- A sufficient condition for DTFT to exist has been shown to be $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$
- This implies that *all stable systems* will have a frequency response

Response of LTI system on a sinusoidal signal

- Let the signal $e^{j\omega_0 n}$ be the input of a stable LTI system whose impulse response is $h(n)$

$$T\{e^{j\omega_0 n}\} = e^{j\omega_0 n} * h(n)$$

$$= \sum_{m=-\infty}^{\infty} h(m) e^{j\omega_0 (n-m)} = e^{j\omega_0 n} \sum_{m=-\infty}^{\infty} h(m) e^{-j\omega_0 m}$$

$$= e^{j\omega_0 n} H(\omega_0) = \left|H(\omega_0)\right| e^{j(\omega_0 n + \Phi(\omega_0))}$$

- The response is also a sinusoidal signal *of the same frequency*
- The magnitude of the response is multiplied by the value of the magnitude response at the frequency ω_0
- The phase of the response is increased by the value of the phase response at the frequency ω_0

Response of LTI system on a sinusoidal signal

- The same holds for real sinusoids as well, since:

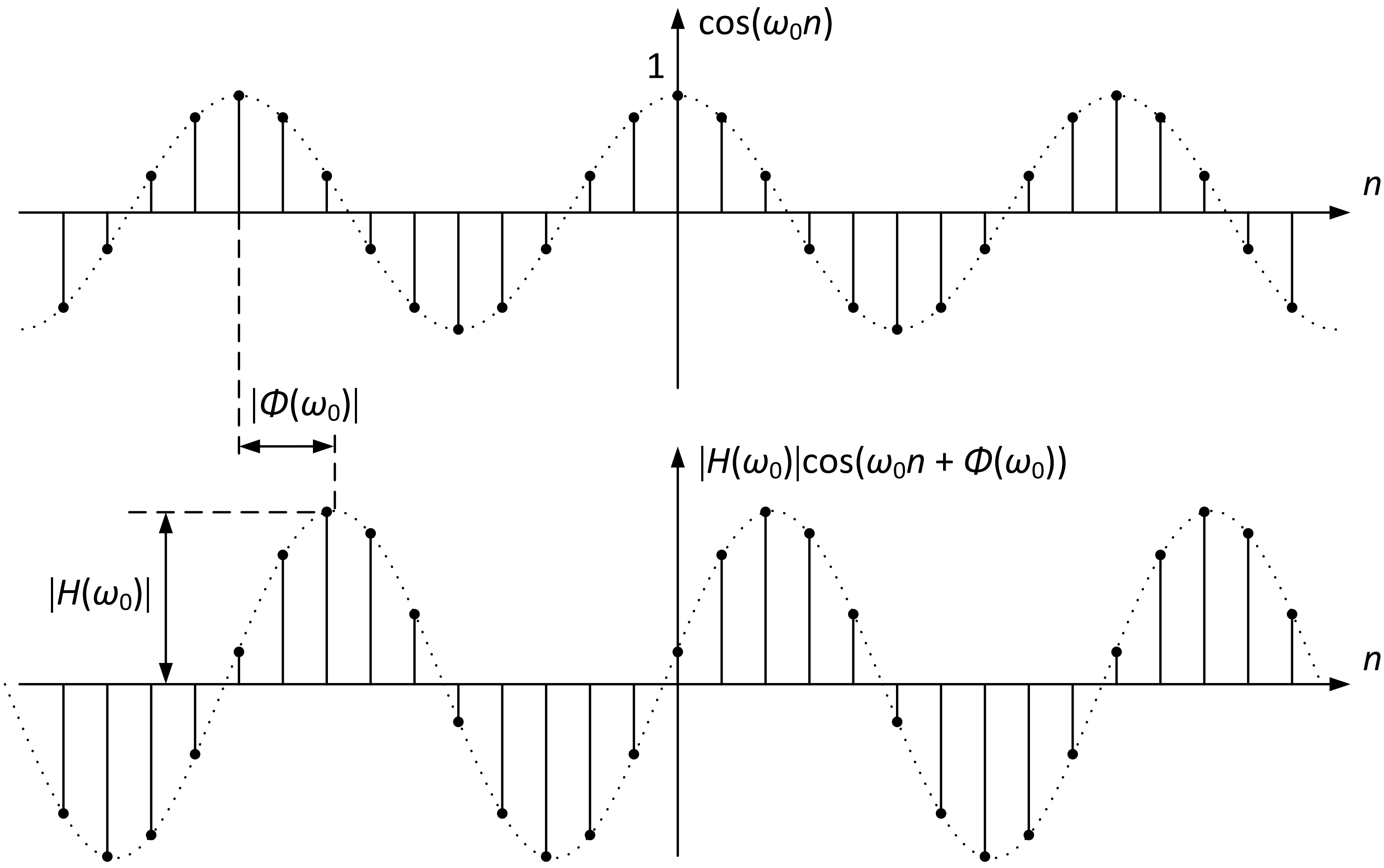
$$\cos(\omega_0 n) = \frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2} \qquad \sin(\omega_0 n) = \frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}$$

- It can be easily shown that:

$$T\{\cos(\omega_0 n)\} = \left|H(\omega_0)\right| \cos(\omega_0 n + \Phi(\omega_0))$$

$$T\{\sin(\omega_0 n)\} = \left|H(\omega_0)\right| \sin(\omega_0 n + \Phi(\omega_0))$$

Response of LTI system on a sinusoidal signal



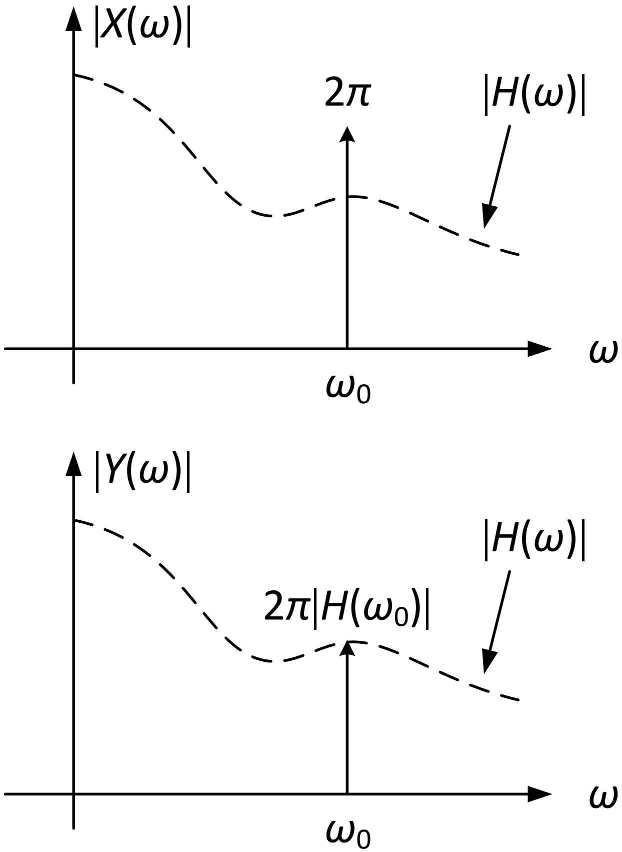
Response of LTI system on a sinusoidal signal

- We can arrive at the same conclusion by looking at things in the frequency domain

$$\mathrm{X}(\omega) = \mathrm{FTD}\{e^{j\omega_0 n}\} = 2\pi\delta_{2\pi}(\omega - \omega_0)$$

$$\begin{aligned} Y(\omega) &= H(\omega)X(\omega) \\ &= H(\omega)2\pi\delta_{2\pi}(\omega-\omega_0) \\ &= H(\omega_0)2\pi\delta_{2\pi}(\omega-\omega_0) \\ &= H(\omega_0)X(\omega) \end{aligned}$$

$$\begin{aligned} y(n) &= \mathrm{IFTD}\{H(\omega_0)\mathrm{X}(\omega)\} = H(\omega_0)e^{j\omega_0 n} \\ &= \left|H(\omega_0)\right|e^{j(\omega_0 n + \Phi(\omega_0))} \end{aligned}$$



Phase delay

- The response to a sinusoid can also be written a little differently:

$$T\{\sin(\omega_0 n)\} = |H(\omega_0)|\sin(\omega_0 n + \Phi(\omega_0))$$
$$= |H(\omega_0)|\sin(\omega_0(n - \tau(\omega_0)))$$

$$\tau(\omega) = -\frac{\Phi(\omega)}{\omega} \quad \text{phase delay}$$

- The phase delay of the system at the frequency ω_0 represents the *time delay* of a sinusoid at this frequency obtained by passing through the system

An example of an FIR system

$$h(n) = \delta(n) + 6\delta(n-1) + 11\delta(n-2) + 6\delta(n-3)$$

$$y(n) = x(n) + 6x(n-1) + 11x(n-2) + 6x(n-3)$$

$$Y(z) = X(z) + 6z^{-1}X(z) + 11z^{-2}X(z) + 6z^{-3}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = 1 + 6z^{-1} + 11z^{-2} + 6z^{-3}$$

$H(z)$ could have been found directly as Z{$h(n)$}.

Note that $z = -1$ is a zero of $H(z)$, which is thus divisible by $(1 + z^{-1})$.

$$H(z) = (1 + z^{-1})(1 + 5z^{-1} + 6z^{-2})$$

$$= (1 + z^{-1})(1 + 2z^{-1})(1 + 3z^{-1})$$

$$H(\omega) = (1 + e^{-j\omega})(1 + 2e^{-j\omega})(1 + 3e^{-j\omega})$$

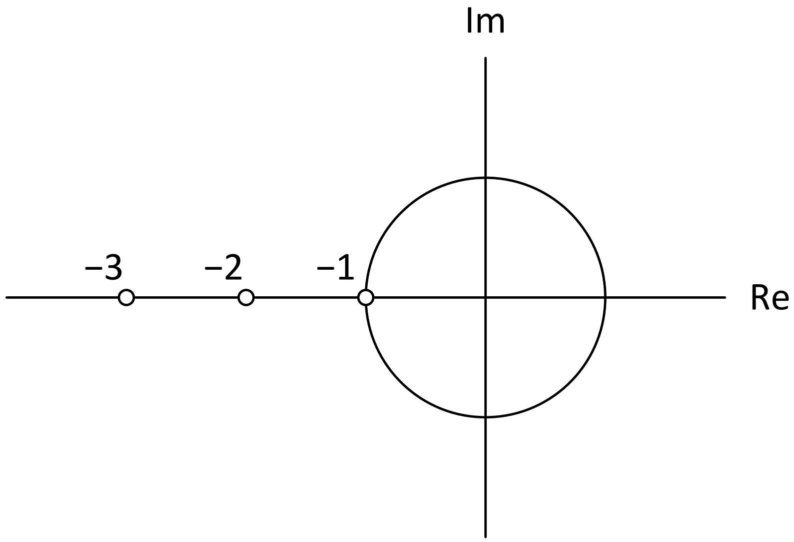
An example of an FIR system

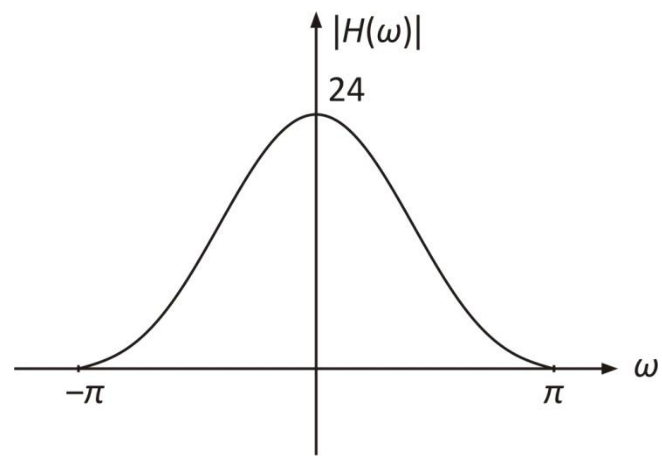
$$H(z) = (1 + z^{-1})(1 + 2z^{-1})(1 + 3z^{-1}) = \frac{(z+1)(z+2)(z+3)}{z^3}$$

Transfer function has 3 zeros, at $z_1 = -1$, $z_2 = -2$, $z_3 = -3$.

$$H(\omega) = (1 + e^{-j\omega})(1 + 2e^{-j\omega})(1 + 3e^{-j\omega})$$

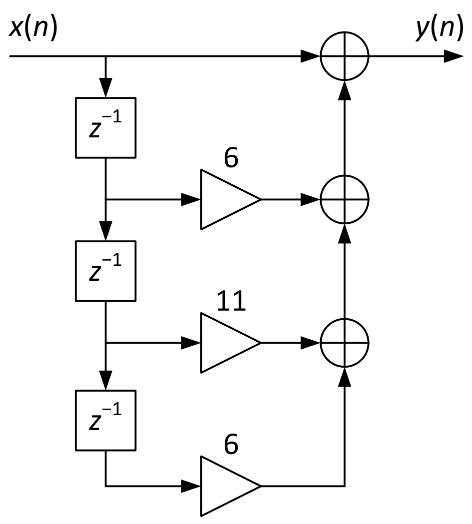
$$|H(\omega)| = |1 + e^{-j\omega}||1 + 2e^{-j\omega}||1 + 3e^{-j\omega}|$$





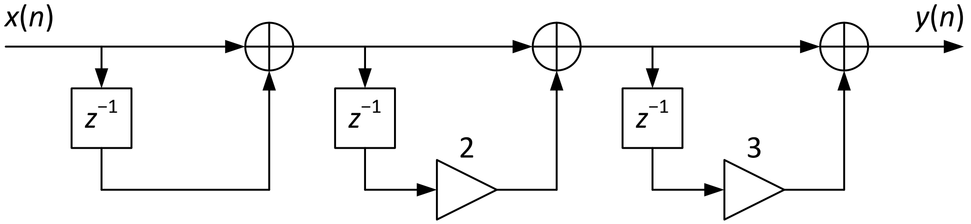
An example of an FIR system

$$H(z) = 1 + 6z^{-1} + 11z^{-2} + 6z^{-3}$$



Direct realization

$$H(z) = (1 + z^{-1})(1 + 2z^{-1})(1 + 3z^{-1})$$



Cascade realization

An example of an FIR system (II)

$$h(n) = \delta(n) - \delta(n-4)$$

$$y(n) = x(n) - x(n-4)$$

$$Y(z) = X(z) - z^{-4}X(z)$$

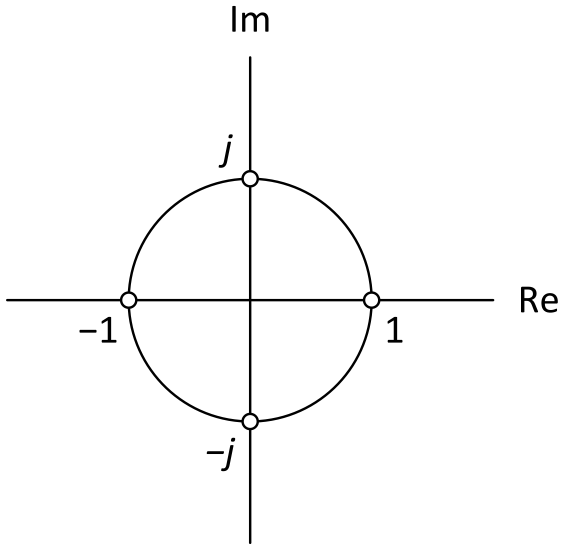
$$H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-4} = \frac{z^4 - 1}{z^4}$$

By equalling the numerator to 0, we obtain:

$$z = e^{j\frac{2k\pi}{4}}, k = 0, 1, 2, 3$$

$$H(z) = \frac{(z-1)(z+1)(z-j)(z+j)}{z^4}$$

$$= (1 - z^{-1})(1 + z^{-1})(1 - jz^{-1})(1 + jz^{-1})$$



An example of an FIR system (II)

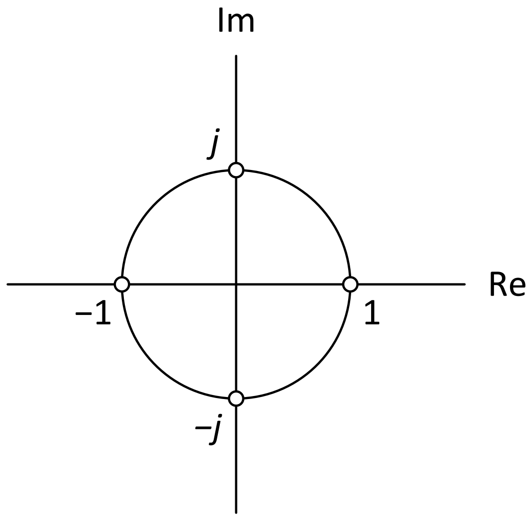
$$H(z) = 1 - z^{-4}$$

$$H(\omega) = 1 - e^{-j4\omega}$$

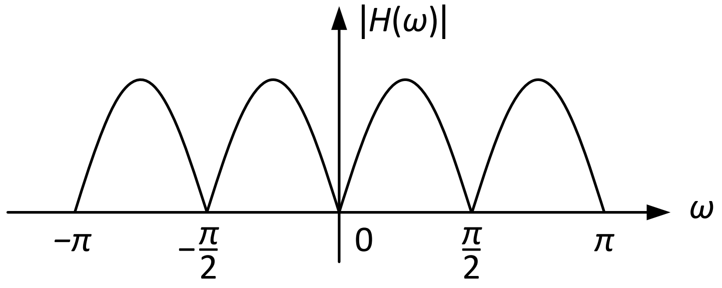
$$= e^{-j2\omega}(e^{j2\omega} - e^{-j2\omega})$$

$$= 2je^{-j2\omega}\sin(2\omega)$$

$$|H(\omega)| = 2|\sin(2\omega)|$$

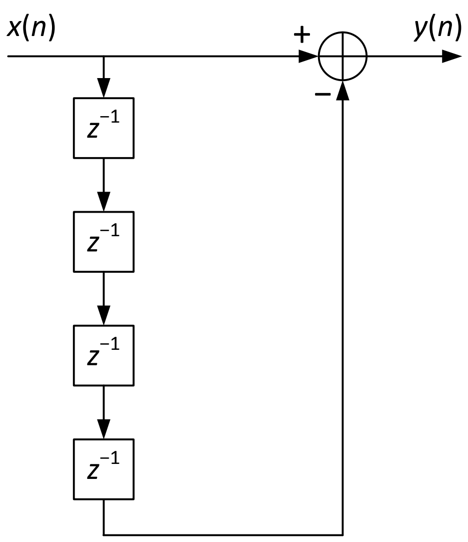


Since the transfer function has zeros at the unit circle, the magnitude response must also have zeros at corresponding frequencies.



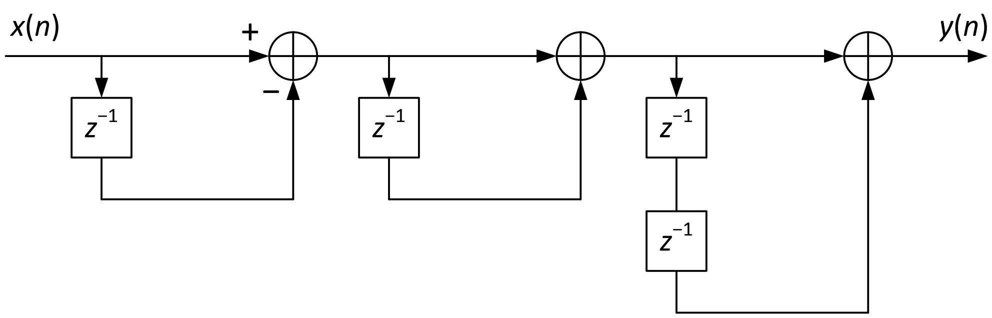
An example of an FIR system (II)

$$H(z) = 1 - z^{-4}$$



Direct realization

$$H(z) = (1 - z^{-1})(1 + z^{-1})(1 - jz^{-1})(1 + jz^{-1})$$
$$= (1 - z^{-1})(1 + z^{-1})(1 + z^{-2})$$



Cascade realization

An example of an IIR system

$$y(n) = -0{,}25y(n-2) + x(n)$$

$$Y(z) = -0{,}25z^{-2}Y(z) + X(z)$$

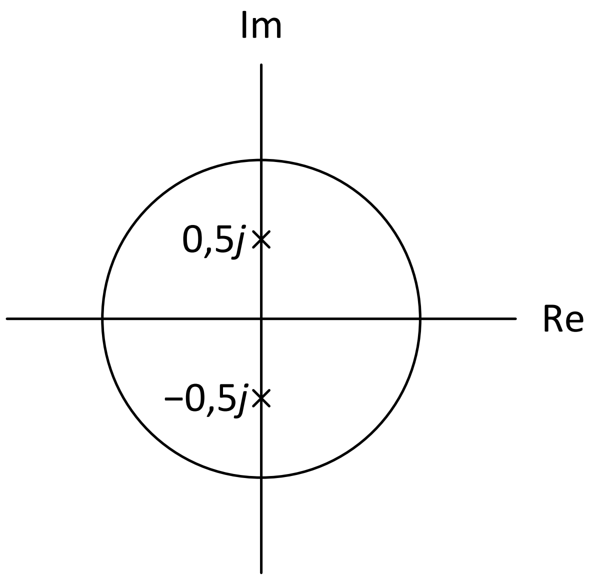
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1+0{,}25z^{-2}} = \frac{z^2}{z^2+0{,}25}$$

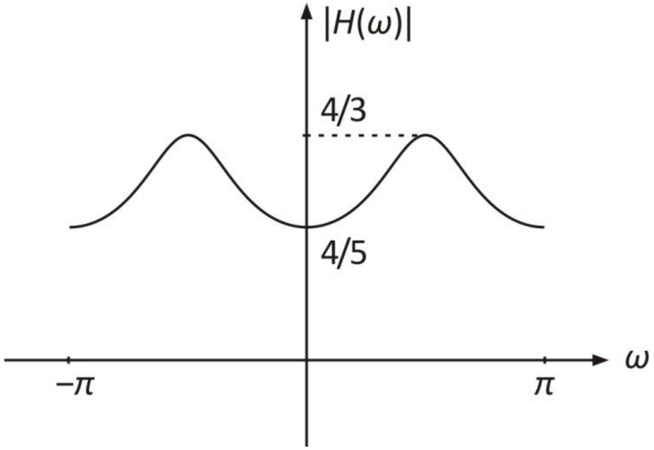
$$H(z) = \frac{0{,}5}{1-j0{,}5z^{-1}} + \frac{0{,}5}{1+j0{,}5z^{-1}}$$

$$h(n) = 0{,}5(0{,}5j)^n u(n) + 0{,}5(-0{,}5j)^n u(n)$$

$$= 0{,}5(0{,}5)^n (e^{jn\pi/2} + e^{-jn\pi/2})u(n)$$

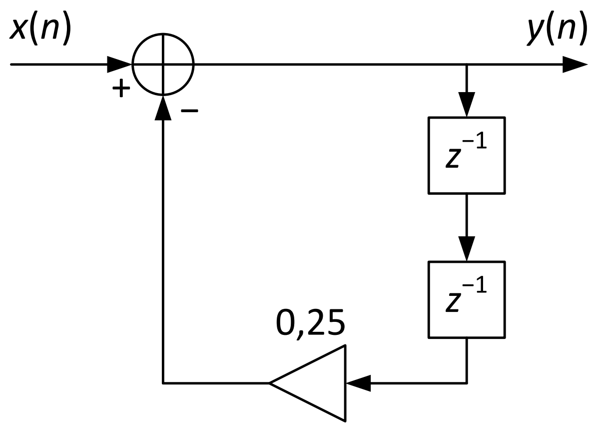
$$= (0{,}5)^n \cos\frac{n\pi}{2} u(n)$$





An example of an IIR system

$$H(z) = \frac{1}{1 + 0{,}25z^{-2}}$$



Direct realization

An example of an IIR system (II)

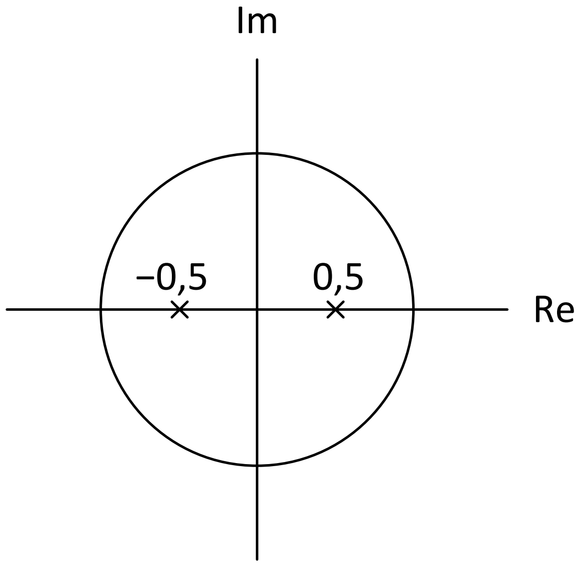
$$y(n) = 0{,}25y(n-2) + x(n)$$

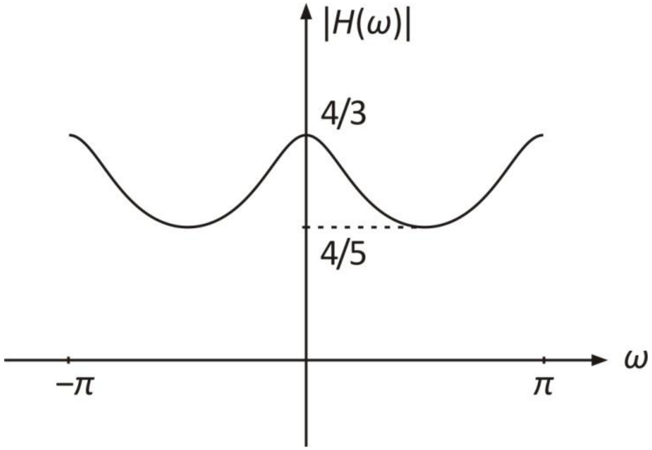
$$Y(z) = 0{,}25z^{-2}Y(z) + X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-0{,}25z^{-2}} = \frac{z^2}{z^2-0{,}25}$$

$$H(z) = \frac{0{,}5}{1-0{,}5z^{-1}} + \frac{0{,}5}{1+0{,}5z^{-1}}$$

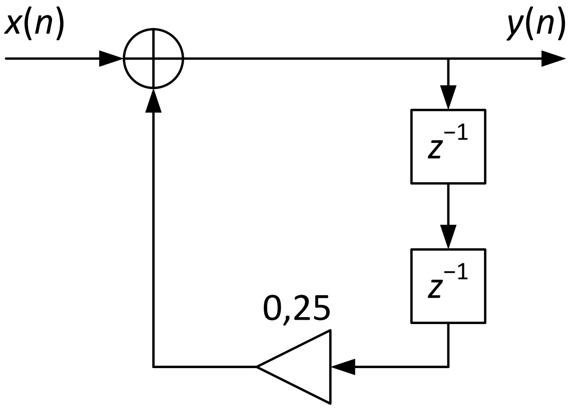
$$h(n) = 0{,}5(0{,}5)^n u(n) + 0{,}5(-0{,}5)^n u(n)$$





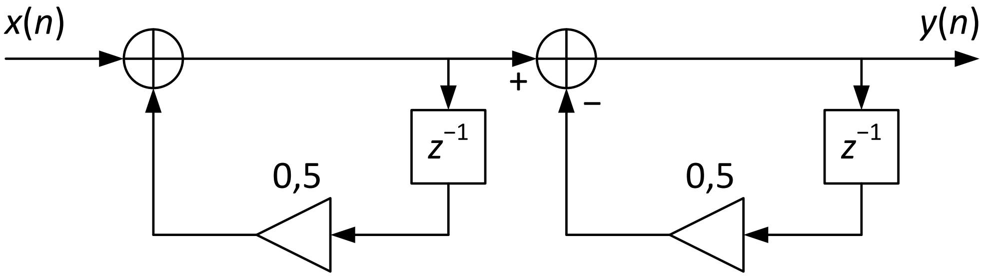
An example of an IIR system (II)

$$H(z) = \frac{1}{1 - 0{,}25z^{-2}}$$



Direct realization

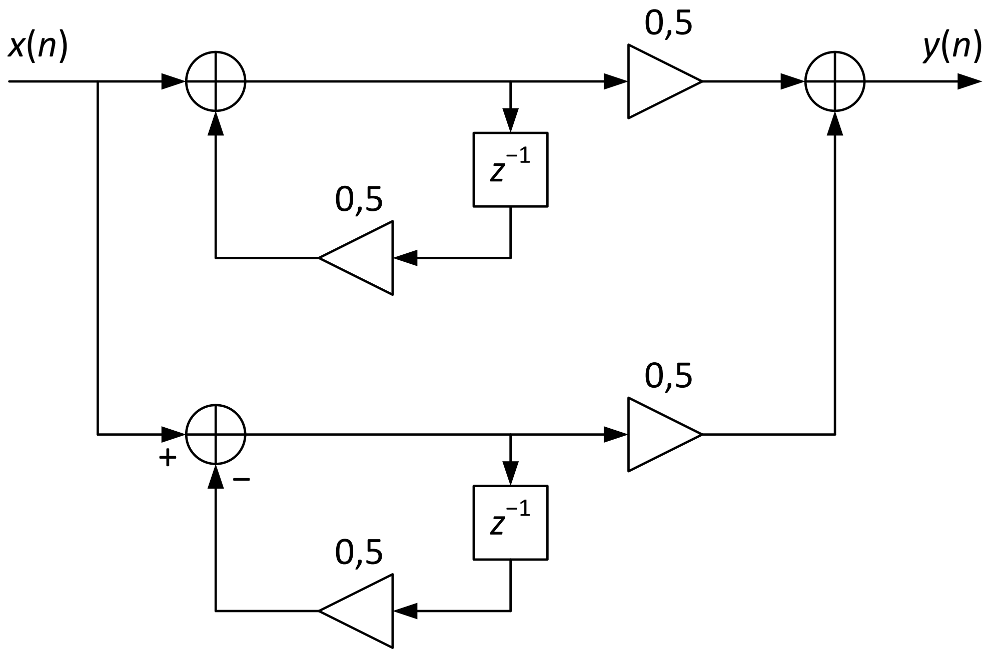
$$H(z) = \frac{1}{1 - 0{,}5z^{-1}} \cdot \frac{1}{1 + 0{,}5z^{-1}}$$



Cascade realization

An example of an IIR system (II)

$$H(z) = \frac{0{,}5}{1 - 0{,}5z^{-1}} + \frac{0{,}5}{1 + 0{,}5z^{-1}}$$



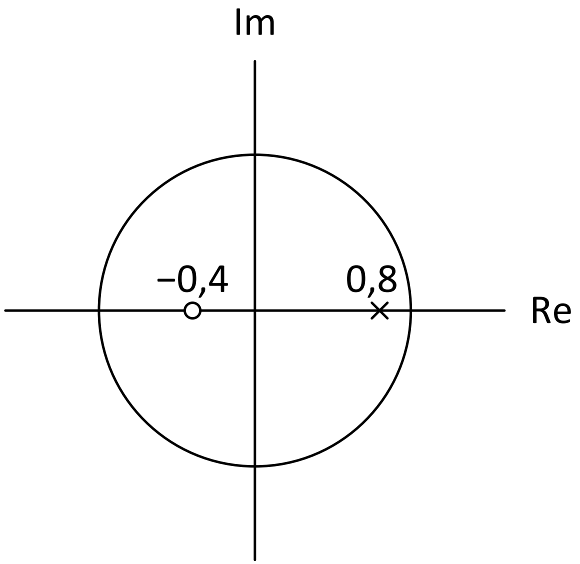
Parallel realization

An example of an IIR system (III)

$$H(z) = \frac{5+2z^{-1}}{1-0,8z^{-1}} = \frac{5}{1-0,8z^{-1}} + \frac{2z^{-1}}{1-0,8z^{-1}}$$

$$h(n) = 5(0,8)^{n} u(n) + 2(0,8)^{n-1} u(n-1)$$

$$= 5\delta(n) + 6(0,8)^{n-1} u(n-1)$$



$$H(z) = \frac{Y(z)}{X(z)} = \frac{5+2z^{-1}}{1-0,8z^{-1}}$$

$$Y(z) - 0,8z^{-1}Y(z) = 5X(z) + 2z^{-1}X(z)$$

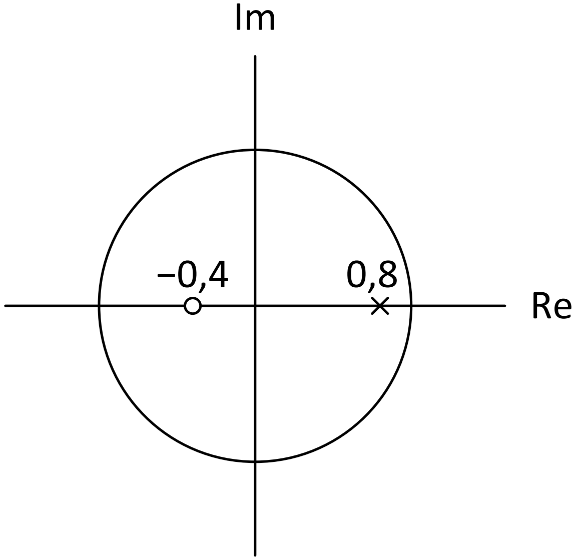
$$y(n) - 0,8y(n-1) = 5x(n) + 2x(n-1)$$

An example of an IIR system (III)

$$H(z)=\frac{5+2z^{-1}}{1-0{,}8z^{-1}}$$

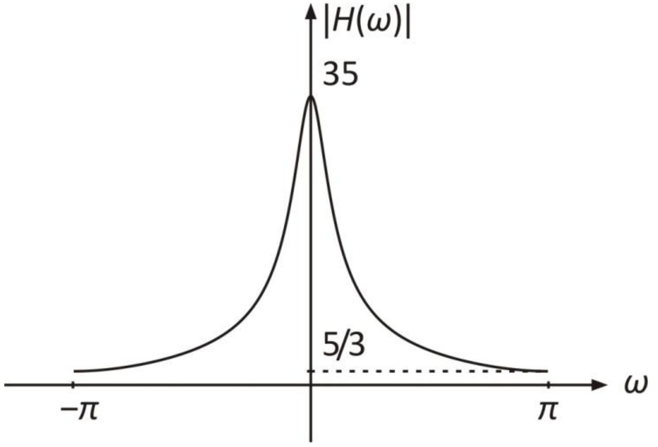
$$H(\omega)=\frac{5+2e^{-j\omega}}{1-0{,}8e^{-j\omega}}=5\cdot\frac{1+0{,}4e^{-j\omega}}{1-0{,}8e^{-j\omega}}$$

$$\left|H(\omega)\right|=5\cdot\frac{\sqrt{1{,}16+0{,}8\cos\omega}}{\sqrt{1{,}64-1{,}6\cos\omega}}$$



$$\left|H(\omega)\right|\Big|_{\omega=0}=5\cdot\frac{1{,}4}{0{,}2}=35$$

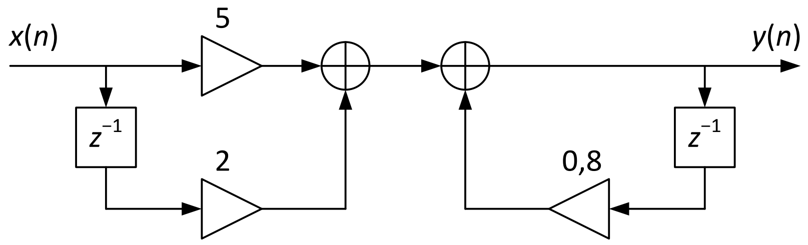
$$\left|H(\omega)\right|\Big|_{\omega=\pi}=5\cdot\frac{0{,}6}{1{,}8}=\frac{5}{3}$$



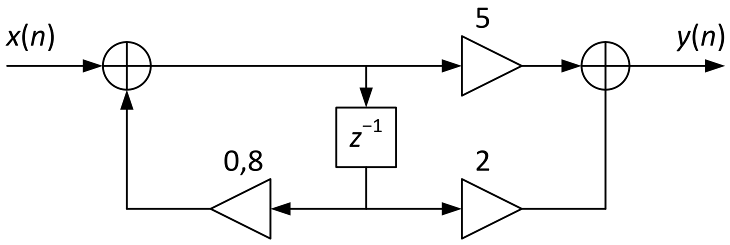
An example of an IIR system (III)

$$H(z) = \frac{5 + 2z^{-1}}{1 - 0{,}8z^{-1}}$$

$$y(n) = 0{,}8y(n-1) + 5x(n) + 2x(n-1)$$



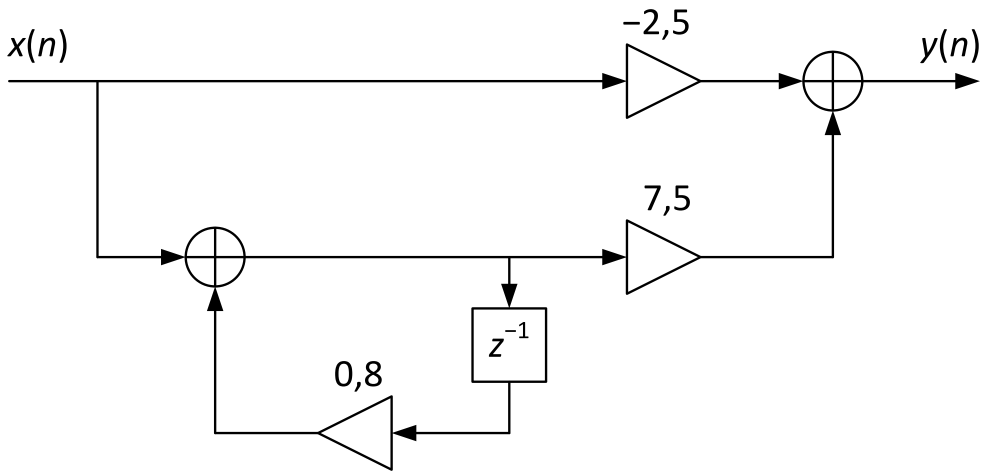
Direct realization



Direct realization (canonical form)

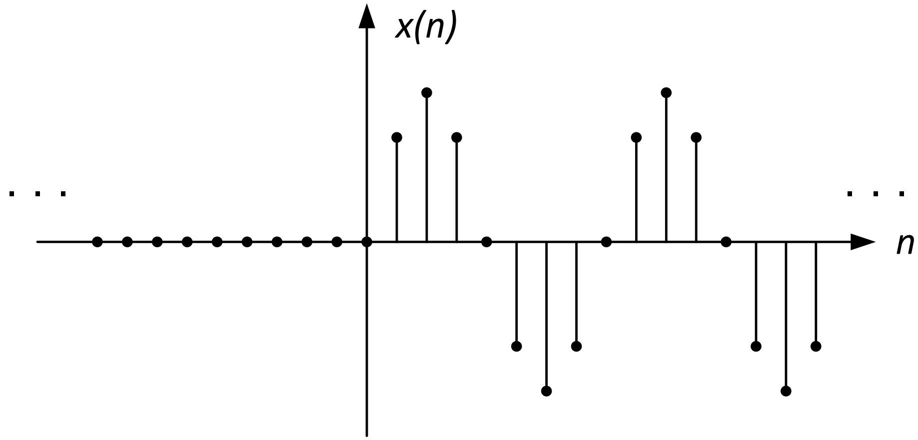
An example of an IIR system (III)

$$H(z) = -2{,}5 + \frac{7{,}5}{1 - 0{,}8z^{-1}}$$



Parallel realization

Response of LTI system on a causal sinusoid



- Of great interest in practice
- Includes two phases:
  - Transient state
  - Steady state
- For a steady state to be reached, the system has to be *stable*

Response of FIR system on a causal sinusoid

$$x(n) = e^{j\omega_0 n} u(n)$$

$$h(n) = h_0\delta(n) + h_1\delta(n-1) + \ldots + h_M\delta(n-M)$$

$$y(n) = \sum_{m=0}^{M} h(m)x(n-m) = \sum_{m=0}^{\min\{n,M\}} h(m)x(n-m)$$

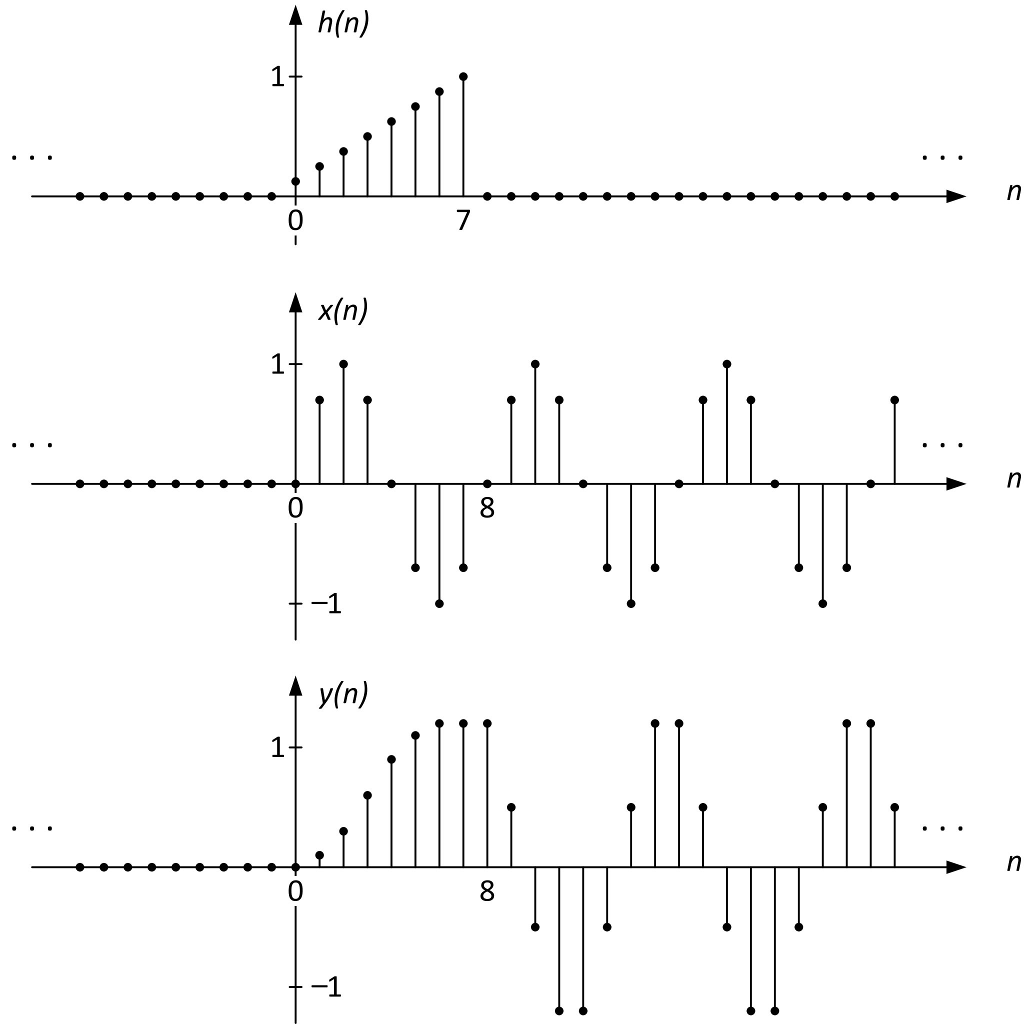
$$= \sum_{m=0}^{\min\{n,M\}} h(m)e^{j\omega_0 (n-m)} = e^{j\omega_0 n} \sum_{m=0}^{\min\{n,M\}} h(m)e^{-j\omega_0 m}$$

$$y(n) = \begin{cases} e^{j\omega_0 n} \sum_{m=0}^{n} h(m)e^{-j\omega_0 m}, \ 0 \le n < M \\ e^{j\omega_0 n} \sum_{m=0}^{M} h(m)e^{-j\omega_0 m} = H(\omega_0)e^{j\omega_0 n}, \ n \ge M \end{cases}$$

Response of FIR system on a causal sinusoid

$$x(n) = \sin\frac{n\pi}{4}u(n)$$

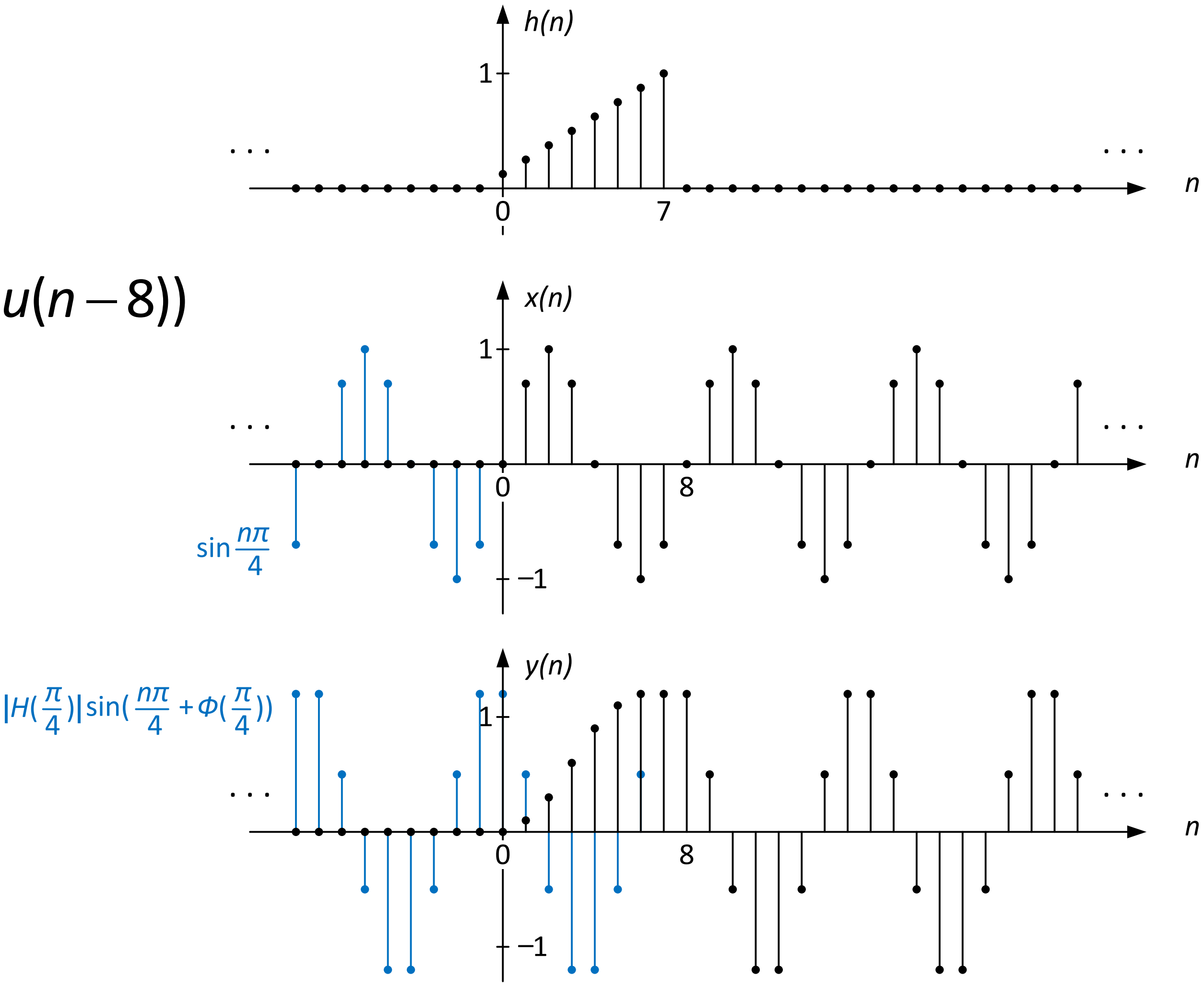
$$h(n) = \frac{1}{8}(n+1)(u(n) - u(n-8))$$



Response of FIR system on a causal sinusoid

$$x(n) = \sin\frac{n\pi}{4}u(n)$$

$$h(n) = \frac{1}{8}(n+1)(u(n)-u(n-8))$$



Response of IIR system on a causal sinusoid

$$x(n) = e^{j\omega_0 n} u(n) \qquad X(z) = \frac{1}{1 - e^{j\omega_0} z^{-1}}, \quad |z| > |e^{j\omega_0}| = 1$$

$$H(z) = \frac{N(z)}{D(z)} = \frac{N(z)}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})\ldots(1 - p_M z^{-1})} \qquad |p_i| < 1$$

$$Y(z) = X(z)H(z) = \frac{A}{1 - e^{j\omega_0} z^{-1}} + \frac{B_1}{1 - p_1 z^{-1}} + \cdots + \frac{B_M}{1 - p_M z^{-1}} \qquad p_i \neq e^{j\omega_0}$$

$$A = (1 - e^{j\omega_0} z^{-1}) Y(z)\Big|_{z = e^{j\omega_0}} = H(z)\Big|_{z = e^{j\omega_0}} = H(\omega_0)$$

$$Y(z) = \frac{H(\omega_0)}{1 - e^{j\omega_0} z^{-1}} + \frac{B_1}{1 - p_1 z^{-1}} + \cdots + \frac{B_M}{1 - p_M z^{-1}} \qquad |z| > 1$$

$$y(n) = H(\omega_0) e^{j\omega_0 n} u(n) + B_1 p_1^n u(n) + \ldots + B_M p_M^n u(n)$$

Response of IIR system on a causal sinusoid

$$y(n) = [H(\omega_0)e^{j\omega_0 n} + B_1 p_1^n + \ldots + B_M p_M^n]u(n)$$

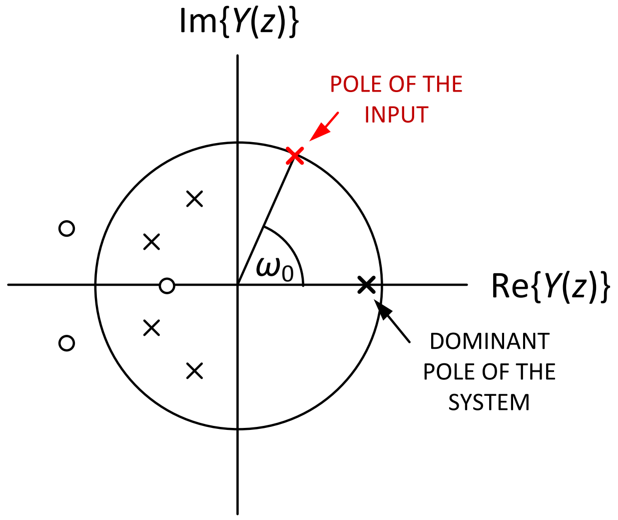
- If for all poles $|p_i|<1$, all terms $B_i p_i^n$ eventually vanish
- Steady state is reached for $n \to \infty$ (in practice, for very large n) and is equal to $H(\omega_0)e^{j\omega_0 n}$, which is exactly the response of LTI systems to a (non-causal) sinusoid
- The rate of convergence to the steady state depends on the positions of the poles, most notably on the position of the pole nearest to the unit circle

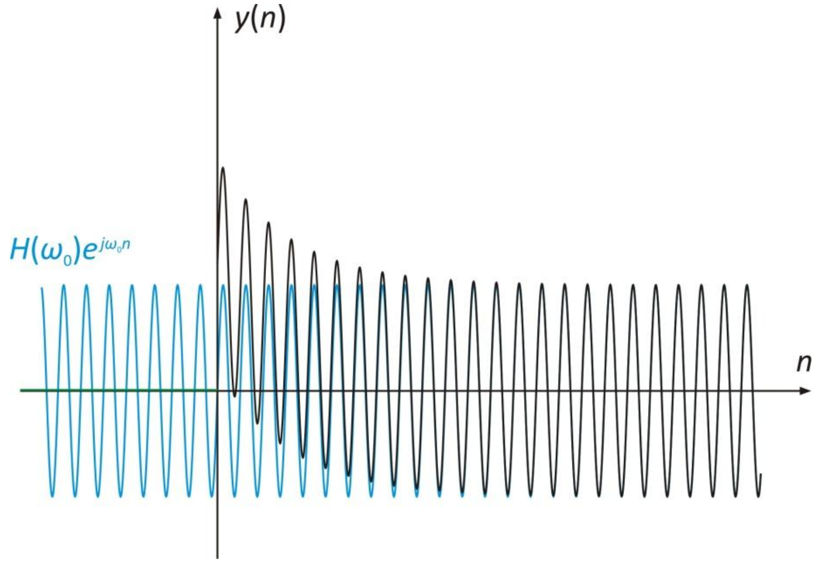
$$\max_i |p_i| = \rho \qquad \rho^{n_{eff}} = \varepsilon \qquad n_{eff} = \frac{\ln \varepsilon}{\ln \rho}$$

Response of IIR system on a causal sinusoid

$$y(n) = [H(\omega_0)e^{j\omega_0 n} + B_1 p_1^n + \ldots + B_M p_M^n]u(n)$$

- If for all poles $|p_i|<1$, all terms $B_i p_i^n$ vanish for sufficiently large n

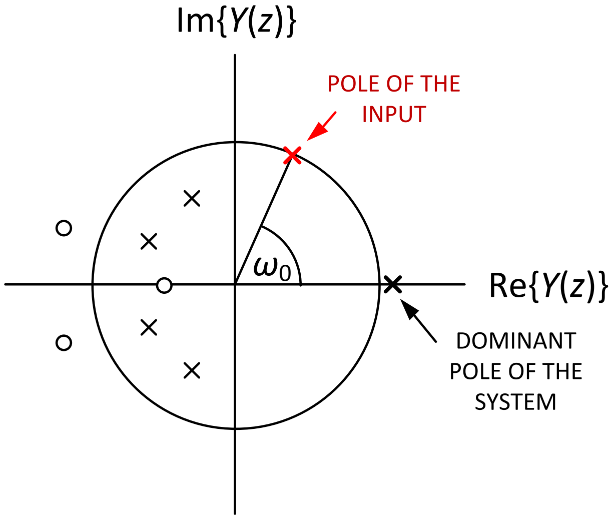


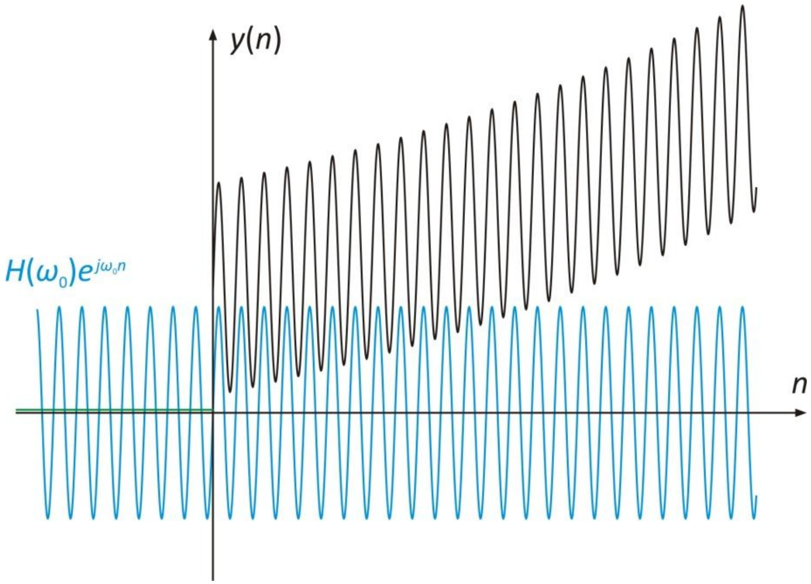


Response of IIR system on a causal sinusoid

$$y(n) = [H(\omega_0)e^{j\omega_0 n} + B_1 p_1^n + \ldots + B_M p_M^n]u(n)$$

- If, for some pole, $|p_i|>1$, $B_i p_i^n$ will go to infinity and there will be no steady state response

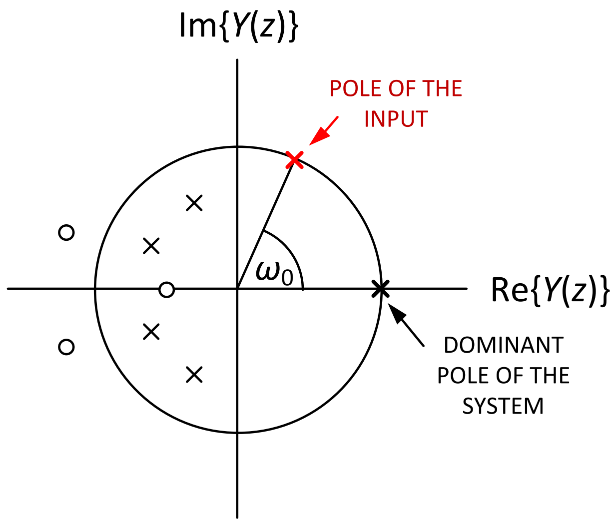


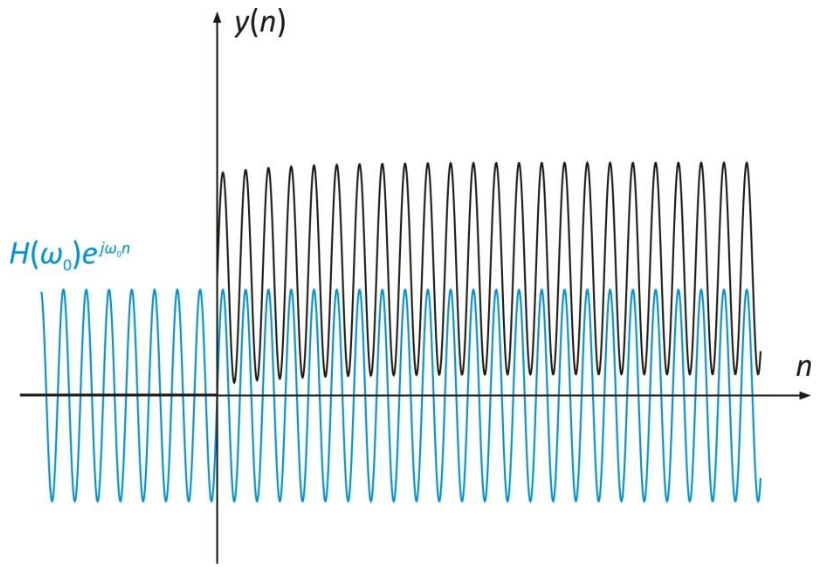


Response of IIR system on a causal sinusoid

$$y(n) = [H(\omega_0)e^{j\omega_0 n} + B_1 p_1^n + \ldots + B_M p_M^n]u(n)$$

- If, for no pole $|p_i|>1$, but there is a simple pole for which $|p_i|=1$, response will contain a component that will not converge to 0





Response of IIR system on a causal sinusoid

- If, for no pole $|p_i|>1$, but there is a simple pole for which $|p_i|=1$, response will contain a component that will not converge to 0
  - If the pole p_i on the unit circle is located at exactly the frequency ω_0, the response will be *unbounded* since $Y(z)$ will contain a double pole!