Z-TRANSFORM

Laplace transform

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{2\pi j} \int_{\gamma - j\infty}^{\gamma + j\infty} X(s) e^{st} ds$$

PIERRE-SIMON LAPLACE
(1749-1827)

- Simplifies the work with continuous-time signals and LTI systems
- Laplace transform of the impulse response of an LTI continuous-time system represents its *transfer function*

Z-transform

- *Z*-transform represents the following mapping of a discrete-time signal into a complex series:

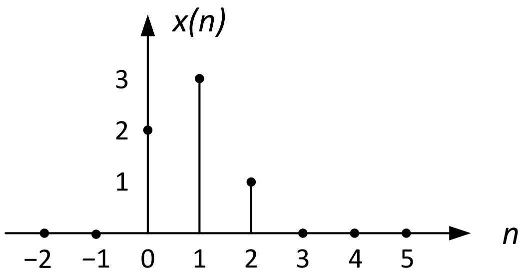
$$X(z) = \mathrm{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(n) = \mathrm{Z}^{-1}\{X(z)\} = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

- There is also *unilateral* *z*-transform:

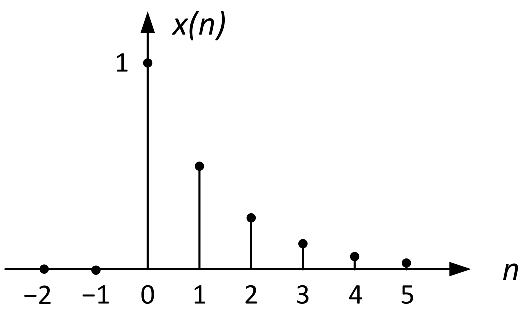
$$X(z) = \mathrm{Z}_u\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

Region of convergence



$$x(n) = 2\delta(n) + 3\delta(n-1) + \delta(n-2)$$

$$X(z) = 2 + 3z^{-1} + z^{-2}, \ z \neq 0$$



$$x(n) = \sum_{k=0}^{\infty} 0{,}5^k \delta(n-k) = 0{,}5^n u(n)$$

$$X(z) = \sum_{n=0}^{\infty} 0{,}5^n z^{-n} = \frac{1}{1 - 0{,}5z^{-1}}, \ |z| > 0{,}5$$

- *The region of convergence (ROC)* is the set of complex values of *z* for which the definition sum of the *z*-transform converges

Rational *z*-transform

- For a wide class of signals *z*-transform can be written as a *rational function* of *z*

$$X(z)=\sum_{n=-\infty}^{\infty}x(n)z^{-n}=\frac{\sum_{i=0}^{M}b_i z^{-i}}{1+\sum_{i=1}^{N}a_i z^{-i}}$$

- When identifying zeros and poles, we assume that *X*(*z*) is irreducible
- *Zeros of ZT* are the roots of the polinomial in the numerator (and ZT, if it exists there, is equal to 0)
- *Poles of ZT* are the roots of the polinomial in the denominator (and ZT, if it exists in the neghbourhood, converges to infinity)
- If the signal is real, zeros and poles are either real or they appear as complex conjugate pairs

Properties of *z*-transform

Linearity

$$Z\{ax(n)+by(n)\}=aX(z)+bY(z)$$

Time shifting

$$Z\{x(n-m)\}=z^{-m}X(z)$$

Scaling in the *z*-domain

$$Z\{a^n x(n)\}=X(a^{-1}z)$$

Properties of *z*-transform

Time reversal

$$Z\{x(-n)\} = X(z^{-1})$$

Differentiation

$$Z\{nx(n)\} = -z\frac{dX(z)}{dz}$$

Transform of convolution

$$Z\{x(n) * y(n)\} = X(z)Y(z)$$

Inverse z-transform

- Inverse z-transform is given by

$$x(n) = Z^{-1}\{X(z)\} = \frac{1}{2\pi j}\oint_C X(z) z^{n-1} dz$$

- C is any counterclockwise closed path encircling the origin and entirely in the region of convergence (ROC)
- The signal $x(n)$ is more easily found using appropriate tables or the residue theory

$$x(n) = \begin{cases} \sum\limits_{p_i\ unutar C} \mathrm{Res}\{X(z)z^{n-1}\}\Big|_{z=p_i}, & n \geq 0 \\ -\sum\limits_{p_i\ van C} \mathrm{Res}\{X(z)z^{n-1}\}\Big|_{z=p_i}, & n < 0 \end{cases}$$

$$\mathrm{Res}\{X(z)z^{n-1}\}\Big|_{z=p_i} = \frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}}(z-p_i)^m X(z) z^{n-1}\right]_{z=p_i}$$

Transfer function

- Z-transform of the impulse response of an LTI system

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$x(n) * h(n) = y(n)$$

$$X(z)H(z) = Y(z)$$

- $H(z)$ can be written in the form of a rational function if and only if the IOR is a linear difference equation with constant coefficients

$$\sum_{i=0}^{M} a_i y(n-i) = \sum_{i=1}^{N} b_i x(n-i)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{M} b_i z^{-i}}{1 + \sum_{i=1}^{N} a_i z^{-i}}$$

Typical shapes of the region of convergence

$$x_1(n) = u(n)$$

$$X_1(z) = \sum_{n=-\infty}^{\infty} u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}$$

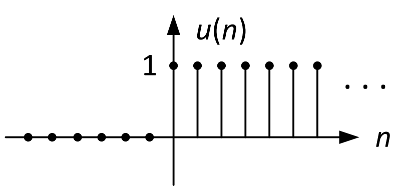
$$= \frac{z}{z-1}, \ |z| > 1$$

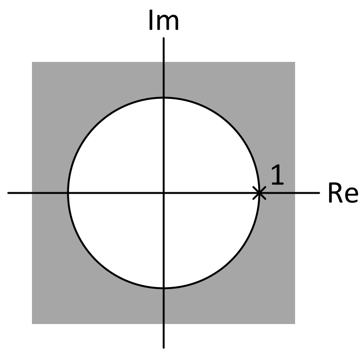
$$x_2(n) = -u(-n-1)$$

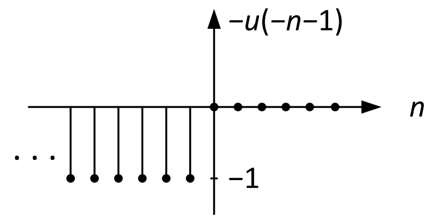
$$X_2(z) = -\sum_{n=-\infty}^{\infty} u(-n-1) z^{-n} = -\sum_{n=-\infty}^{-1} z^{-n}$$

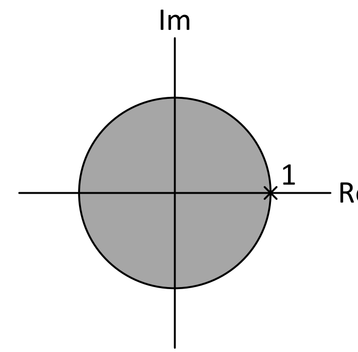
$$= -\sum_{n=1}^{\infty} z^{n} = 1 - \sum_{n=0}^{\infty} z^{n}$$

$$= 1 - \frac{1}{1-z} = \frac{z}{z-1}, \ |z| < 1$$









Typical shapes of the region of convergence

- Finite duration signals
  - Infinite sum becomes finite:

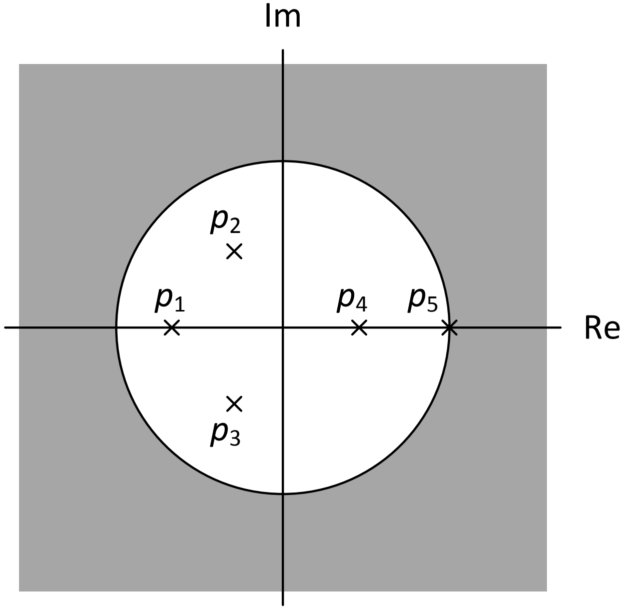
$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=N_1}^{N_2} x(n) z^{-n}$$

  - This sum may not converge only for $z = 0$ or $z \to \infty$, thus ROC is (almost) the entire *z*-plane
  - These poles are considered trivial, as their influence on the behaviour of the system is limited to shifting the response in time

Typical shapes of the region of convergence

- Causal infinite duration signals

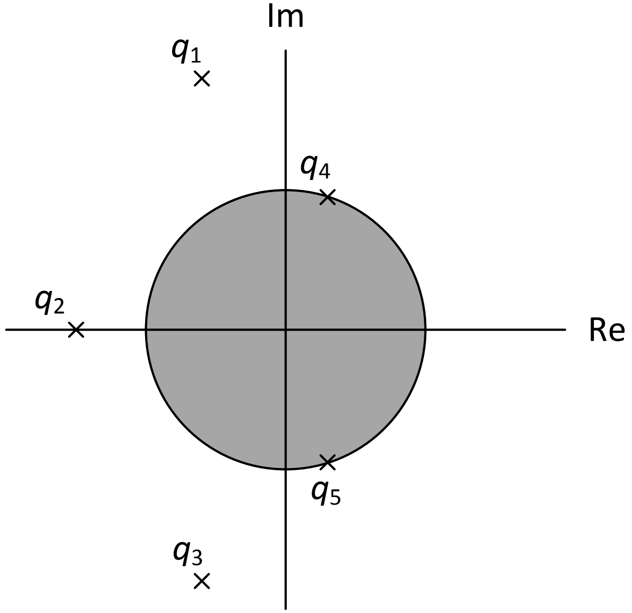
$$x(n) = A_1 p_1^n u(n) + A_2 p_2^n u(n) + \ldots + A_N p_N^n u(n)$$



Typical shapes of the region of convergence

- Anticausal infinite duration signals

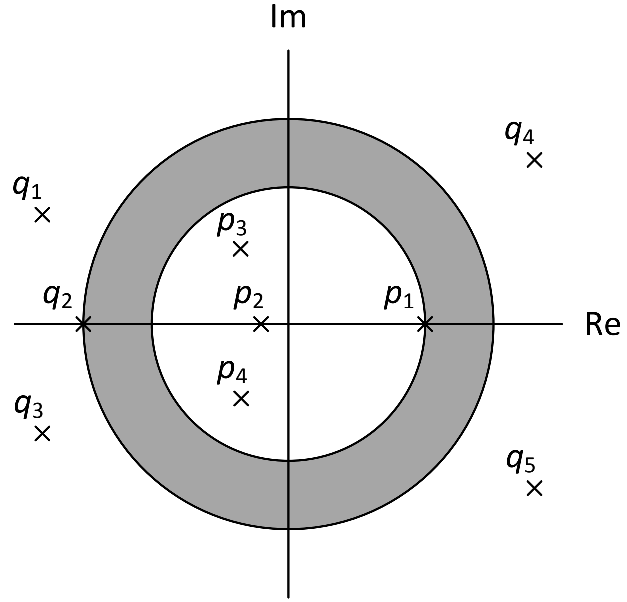
$$x(n) = B_1 q_1^n u(-n-1) + B_2 q_2^n u(-n-1) + \ldots + B_M q_M^n u(-n-1)$$



Typical shapes of the region of convergence

- General case

$$x(n) = A_1 p_1^n u(n) + A_2 p_2^n u(n) + \ldots + A_N p_N^n u(n)$$
$$+ B_1 q_1^n u(-n-1) + B_2 q_2^n u(-n-1) + \ldots + B_M q_M^n u(-n-1)$$



Stability

- Any system is stable if the output signal $y(n)$ remains bounded for each bounded input signal $x(n)$
  - This is referred to as BIBO (Bounded Input Bounded Output) stability
- LTI system is stable
  - if and only if its impulse response is absolutely summable

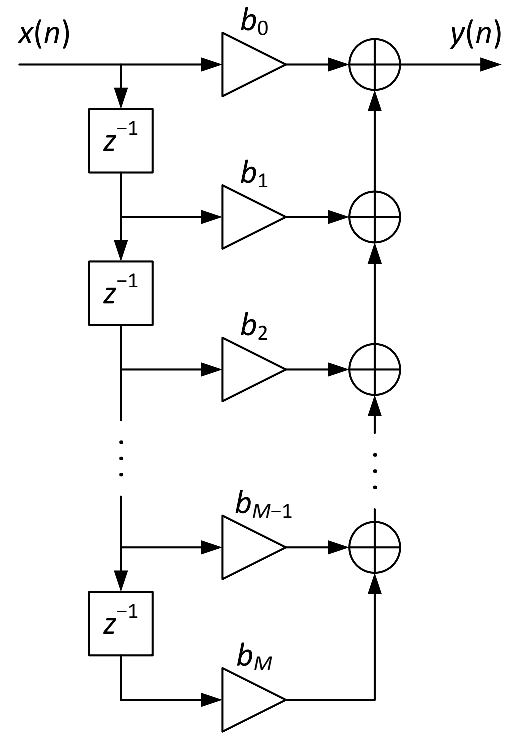
$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

    which obviously implies that the impulse response converges to 0 when $n \to \pm\infty$
  - if and only if the unit circle in the *z*-plain lies in the region of convergence

Stability

- FIR systems
  - Impulse response is absolutely summable, so they are *always stable*
  - Canonical block diagram of realization contains no feedback (recursive) loops
  - Irreducible transfer function contains no non-trivial poles

$$y(n) = \sum_{i=0}^{M} b_i x(n-i)$$



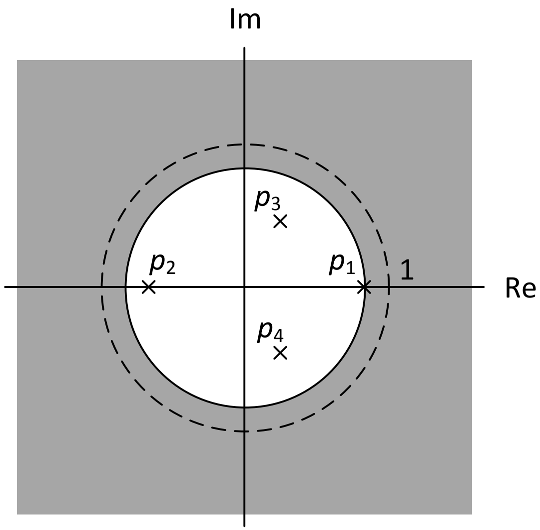
$$H(z) = \frac{\sum_{i=0}^{M} b_i z^{-i}}{1 + \sum_{i=1}^{N} a_i z^{-i}} = \sum_{i=0}^{M} b_i z^{-i}$$

0

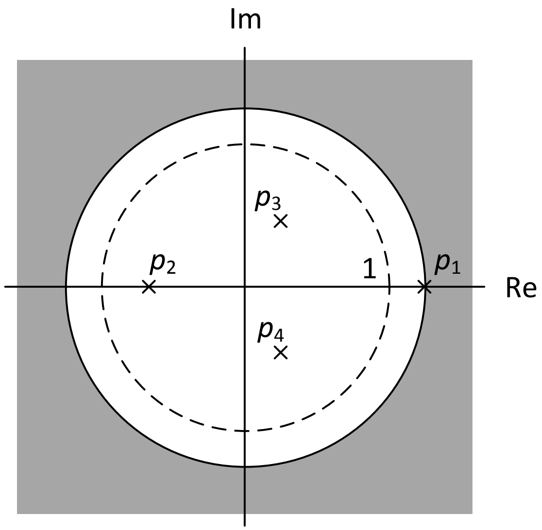
Stability

- Causal IIR systems

$$h(n) = A_1 p_1^n u(n) + A_2 p_2^n u(n) + \ldots + A_N p_N^n u(n)$$



stable

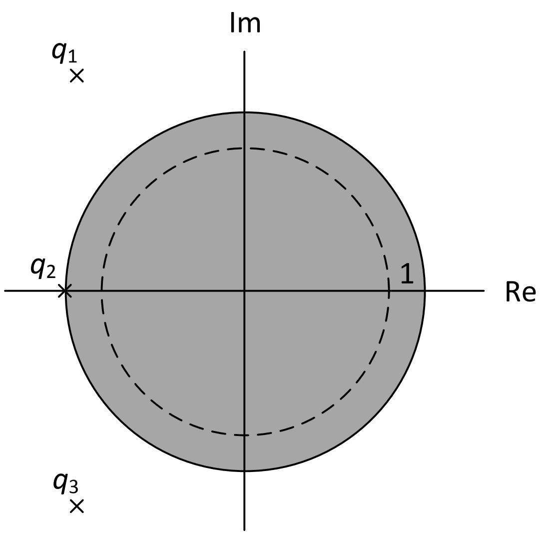


unstable

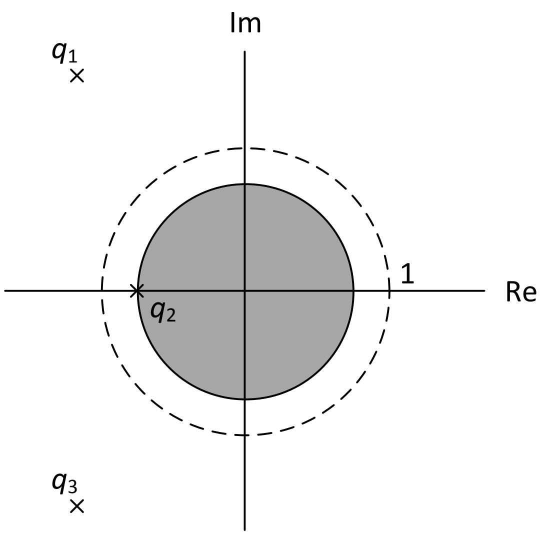
Stability

- Anticausal IIR systems

$$h(n) = B_1 q_1^n u(-n-1) + B_2 q_2^n u(-n-1) + \ldots + B_M q_M^n u(-n-1)$$



stable

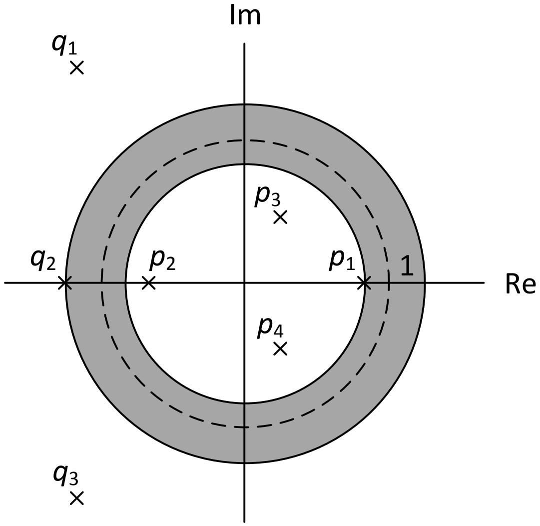


unstable

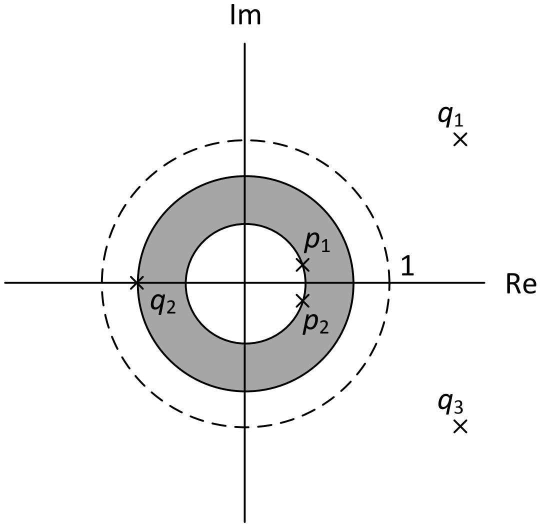
Stability

- Typical general case

$$h(n) = A_1 p_1^n u(n) + A_2 p_2^n u(n) + \ldots + A_N p_N^n u(n)$$
$$+ B_1 q_1^n u(-n-1) + B_2 q_2^n u(-n-1) + \ldots + B_M q_M^n u(-n-1)$$



stable



unstable

Marginal stability

- The unit circle represents the boundary of the region of convergence and all poles on the unit circle are distinct
- Marginally stable systems are not BIBO stable
  - There is a bounded input signal which will result in unbounded output
  - For an impulse input, the output will be bounded but it will not return to zero
- An example of such a system is the LTI system with impulse response $h(n)=u(n)$