VEROVATNOĆA, 16.6.2017.

1. A - slučajno izabrani student uči Francuski

   B - slučajno — II — uči Nemački

$$\underline{P(A|B) = ?} \qquad P(A|B) = \frac{P(AB)}{P(B)}$$

$$P(A) = 0.15, \quad P(B) = 0.35$$

$$P(A \cup B) = 0.4$$

$P(AB) = ?$

$$P(A \cup B) = P(A) + P(B) - P(AB)$$

$$P(AB) = -0.4 + 0.15 + 0.35 = 0.10$$

$$P(A|B) = \frac{0.10}{0.35} = \frac{2}{7}$$

(2.) X, Y nezavisne, $\mathcal{E}(\mu)$ raspodela

$$\varphi_X(x) = \begin{cases} \mu e^{-\mu x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

$$V = X+Y, \quad W = \frac{X}{X+Y}$$

$$\varphi_Y(y) = \begin{cases} \mu e^{-\mu y}, & y > 0 \\ 0, & y \leq 0 \end{cases}$$

$\varphi_{(V,W)}(v,w) = ?$

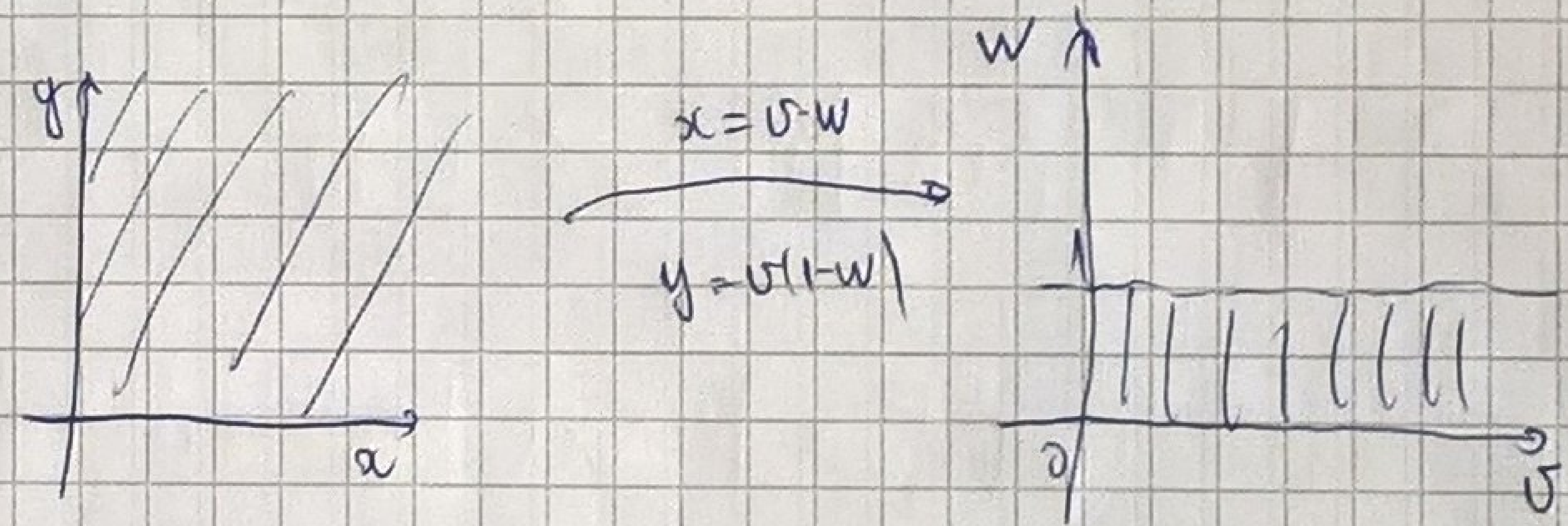
$$v = x+y$$

$$w = \frac{x}{x+y} \Rightarrow x = v \cdot w$$

$$y = v - vw = v(1-w)$$

$$J(v,w) = \begin{vmatrix} w & v \\ 1-w & -v \end{vmatrix} = -v$$

Dakle, $\varphi_{(V,W)}(v,w) = \mu e^{-\mu v w} \cdot \mu e^{-\mu v(1-w)} \cdot |-v| = \mu^2 v e^{-\mu v}, \; v>0, 0<w<1$



$\varphi_{(V,W)}(v,w) = 0$, inače

$$\varphi_V(v) = \int_0^1 \mu^2 v e^{-\mu v} \, dw = \mu^2 v e^{-\mu v}, \quad v>0$$

$\varphi_V(v) = 0$, inače

$$\varphi_W(w) = \int_0^\infty \mu^2 v e^{-\mu v} dv = 1 \quad \text{za} \quad 0<w<1$$

$\varphi_W(w) = 0$ inače

Dalje je $\varphi_{(V,W)}(v,w) = \varphi_V(v)\, \varphi_W(w) \Rightarrow$ V i W su nezavisne

(5.) $N : \mathcal{N}(0,1)$ $\quad x^2 + 2Nx + 1 = 0$

$$x_{1,2} = \frac{-2N \pm \sqrt{4N^2-4}}{2} = \frac{-2N \pm 2\sqrt{N^2-1}}{2}$$

$$x_1 + x_2 = -N + \sqrt{N^2-1} + (-N - \sqrt{N^2-1}) = -2N$$

Dakle, $\underline{S = -2N}$

Korijeni jednadžbe $x^2+2Nx+1=0$ su realni samo ako je $N^2-1 \geq 0$, to jest $\underline{|N| \geq 1}$

Dakle, prvo treba odrediti funkciju razdiobe sluč. var. $S=-2N$, ako je $|N| \geq 1$, odnosno

$$P(-2N < x \mid |N| \geq 1) = ? \quad (= F_{-2N \mid |N| \geq 1}(x))$$

$$P(-2N < x \mid |N| \geq 1) = \frac{P(-2N < x, |N| \geq 1)}{P(|N| \geq 1)}$$

$$P(|N| \geq 1) = P(N \leq -1) + P(N \geq 1)$$

$$P(N \leq -1) = \Phi(-1) - \Phi(-\infty) = -\Phi(1) - \left(-\frac{1}{2}\right)$$

$$\underline{P(N \geq 1) = -\Phi(1) + \Phi(\infty) = \frac{1}{2} - \Phi(1)}$$

$$\Rightarrow P(|N| \geq 1) = -\Phi(1) + \frac{1}{2} + \frac{1}{2} - \Phi(1) = 1 - 2\Phi(1)$$

Dalje, $P(-2N < x, |N| \geq 1) = P(N > -\frac{1}{2}x, |N| \geq 1) = P(|N| \geq 1)$

za $x \in [-2, 2)$

$$P(N > -\tfrac{1}{2}x, |N| \geq 1) = \begin{cases} P(N > -\frac{1}{2}x) = 1 - \Phi(-\frac{1}{2}x), & x \leq -2 \\ P(-\frac{1}{2}x \leq N \leq -1) = \Phi(-1) - \Phi(-\frac{1}{2}x), & x \geq 2 \end{cases}$$

Zatim, $\varphi_{-2N \mid |N| \geq 1}(x) = F'(x)$, tamo gdje je F diferencijabilna

(4) $(X_n)_{n\in\mathbb{N}}$, $X_n : \mathcal{U}(0,n)$, $n=1,2,\dots$ $\rightarrow$ $\varphi_{X_n}(x) = \begin{cases} \frac{1}{n}, & x\in(0,n) \\ 0, & \text{inače} \end{cases}$

$$Y_n = \frac{X_n}{1-X_n}$$

$$\begin{array}{ccccc} \boxed{\text{s.s.}} & \longrightarrow & \boxed{\text{u ver.}} & \longrightarrow & \boxed{\text{u raspodeli}} \\ & & \uparrow & & \\ & & \boxed{\text{s.k.}} & & \end{array}$$

$$E(Y_n) = \int_{-\infty}^{\infty} \frac{x}{1-x}\varphi_{X_n}(x)\,dx = \int_0^n \frac{x}{1-x}\frac{1}{n}\,dx = \frac{1}{n}\int_0^n \frac{x}{1-x}\,dx \rightarrow \text{integral divergira!}$$

Međutim, $Y_n = \dfrac{X_n}{1-X_n} = \dots = -1 - \dfrac{1}{X_n-1}$, pa ćemo ispitivati konvergenciju ka -1

Da li $Y_n \longrightarrow -1$ u verovatnoći?

$$P\{|Y_n+1|\geq\varepsilon\} = P\left\{\left|\frac{X_n}{1-X_n}+1\right|\geq\varepsilon\right\} = P\left\{\left|\frac{X_n+1-X_n}{1-X_n}\right|\geq\varepsilon\right\}$$

$$= P\left\{\left|\frac{1}{1-X_n}\right|\geq\varepsilon\right\} = P\left\{|X_n-1|\leq\frac{1}{\varepsilon}\right\} = P\left\{-\frac{1}{\varepsilon}\leq X_n-1\leq\frac{1}{\varepsilon}\right\} =$$

$$= P\left\{1-\frac{1}{\varepsilon}\leq X_n\leq 1+\frac{1}{\varepsilon}\right\} = F_{X_n}\left(1+\frac{1}{\varepsilon}\right) - F_{X_n}\left(1-\frac{1}{\varepsilon}\right), \text{ gde je}$$

$$F_{X_n}(x) = \begin{cases} 0, & x\leq 0 \\ \frac{x}{n}, & 0<x\leq n \\ 1, & x>n \end{cases} \quad (\text{znamo da } X_n : \mathcal{U}(0,n))$$

1) $\varepsilon\leq 1 \rightarrow 1-\frac{1}{\varepsilon}\leq 0 \rightarrow F_{X_n}(1-\varepsilon)=0$

Tada je za dovoljno veliko n:

$$P\{|Y_n+1|\geq\varepsilon\} = F_{X_n}\left(1+\frac{1}{\varepsilon}\right) = \frac{\varepsilon+1}{n\varepsilon}$$

2) $\varepsilon>1$

Tada je za dovoljno veliko n:

$$P\{|Y_n+1|\geq\varepsilon\} = F_{X_n}\left(1+\frac{1}{\varepsilon}\right) - F_{X_n}\left(1-\frac{1}{\varepsilon}\right) = \frac{\varepsilon+1}{n\varepsilon} - \frac{\varepsilon-1}{n\varepsilon} = \frac{2}{n\varepsilon}$$

Dakle, $P\{|Y_n+1|\geq \varepsilon\} = \begin{cases} \frac{1+\varepsilon}{n\varepsilon}, & \frac{1}{n-1} < \varepsilon \leq 1 \\ \frac{2}{n\varepsilon}, & \varepsilon > 1 \end{cases} \xrightarrow[n\to\infty]{} 0$, za sve $\varepsilon > 0$

Sledi da $Y_n \xrightarrow{V} -1$, pa $Y_n \to -1$ i u raspodeli

Ako konvergira s.s., onda granica mora biti -1, pošto s.s. implicira konvergenciju u verovatnoći. Međutim, $\sum_{n=n_0}^{\infty} P\{|Y_n+1|\geq\varepsilon\} = \infty$, $\forall \varepsilon > 0$

$\Rightarrow$ niz (Y_n) ne konvergira s.s.

Kako je $E(Y_n^2) = \int_{-\infty}^{\infty} \frac{x^2}{(1-x)^2}\varphi_{X_n}(x)dx = \frac{1}{n}\int_0^n \frac{x^2}{(1-x)^2}dx \to$ divergira

$\Rightarrow$ nema smisla pričati o konvergenciji u srednje kvadratnom

(5) $X: \mathcal{E}(\frac{1}{20})$, X-vreme presijecanja bazena

$E(X) = 20$, $D(X) = 400$

X_i - količina vode koja isteknu u i-tom satu $i=1,2,\dots,12$

$X_1, X_2, \dots, X_{12}$ su nezavisne i

$E(X_i) = 20$, $D(X_i) = 400$

Količina vode koja istekne za 12 sati: $Y_{12} = \sum_{i=1}^{12} X_i$

Treba odrediti $P(Y_{12} \geq 580)$

$E(Y_{12}) = 12 \cdot 20 = 240$, $D(Y_{12}) = 4800$

CGT: $P(Y_{12} \geq 580) = P\left(\frac{Y_{12}-240}{\sqrt{4800}} > \frac{580-240}{\sqrt{4800}}\right)$

$P(Y_{12} \geq 580) \approx \Phi(\infty) - \Phi(\dots)$