Višedimenzionalna normalna raspodela

$X = (X_1, \dots, X_n) : \mathcal{N}(m, V)$

$V \dots$ simetrična, pozitivno definitna $n \times n$ matrica

$m = (m_1, \dots, m_n) \dots$ fiksiran vektor kolona

Tada je

$$\varphi_X(x_1, \dots, x_n) = \frac{1}{\sqrt{(2\pi)^n}\sqrt{\det V}} e^{-\frac{1}{2}(x-m)^T V^{-1}(x-m)}$$

Specijalno, za $n=2$

$$V = \begin{bmatrix} \sigma_1^2 & \rho \\ \rho & \sigma_2^2 \end{bmatrix}, \quad m = (m_1, m_2), \quad \sigma_1, \sigma_2 > 0, \quad \rho \in (-1,1)$$

imamo

$$\varphi_{(X_1,X_2)}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-m_1}{\sigma_1}\right)^2 - 2\rho\frac{x_1-m_1}{\sigma_1}\frac{x_2-m_2}{\sigma_2} + \left(\frac{x_2-m_2}{\sigma_2}\right)^2\right]}$$

Pišemo $(X_1,X_2) : \mathcal{N}(m_1, m_2, \sigma_1^2, \sigma_2^2, \rho)$

Teorema. Ako $(X_1, X_2) : \mathcal{N}(m_1, m_2, \sigma_1^2, \sigma_2^2, \rho)$, onda

$X_i : \mathcal{N}(m_i, \sigma_i^2)$, $i = 1,2$.

(Svaka komponenta vektora sa Normalnom raspodelom takođe ima Normalnu raspodelu)

dokaz. Dokazaćemo da $X_1 : \mathcal{N}(m_1, \sigma_1^2)$. Za X_2 dokaz ide analogno.

Treba pokazati $\varphi_{X_1}(x_1) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x_1-m_1)^2}{2\sigma_1^2}}$, $x_1 \in \mathbb{R}$

Neka je $x_1 \in \mathbb{R}$ proizvoljno.

$$\varphi_{X_1}(x_1) = \int_{-\infty}^{\infty} \varphi_{(X_1,X_2)}(x_1,x_2)\,dx_2 =$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-m_1}{\sigma_1}\right)^2 - 2\rho\frac{x_1-m_1}{\sigma_1}\frac{x_2-m_2}{\sigma_2} + \left(\frac{x_2-m_2}{\sigma_2}\right)^2\right]} dx_2$$

smena: $t = \frac{x_2 - m_2}{\sigma_2} \Rightarrow dt = \frac{1}{\sigma_2} dx_2$

$x_2 = -\infty \rightarrow t = -\infty$

$x_2 = +\infty \rightarrow t = +\infty$

$$= \frac{1}{2\pi\sigma_1\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-m_1}{\sigma_1}\right)^2 - 2\rho t \frac{x_1-m_1}{\sigma_1} + t^2\right]} dt$$

$$= \frac{1}{2\pi\sigma_1\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\underbrace{t-\rho\frac{x_1-m_1}{\sigma_1}}_{=v \ \Rightarrow\ dt=dv}\right)^2 + (1-\rho^2)\left(\frac{x_1-m_1}{\sigma_1}\right)^2\right]} dt$$

$$= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x_1-m_1)^2}{2\sigma_1^2}} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{v}{\sqrt{1-\rho^2}}\right)^2} dv$$

smena: $u = \frac{v}{\sqrt{1-\rho^2}} \Rightarrow du = \frac{1}{\sqrt{1-\rho^2}} dv$

$$= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x_1-m_1)^2}{2\sigma_1^2}} \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du}_{=\sqrt{2\pi} \text{ (Poasonov integral)}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x_1-m_1)^2}{2\sigma_1^2}}$$

Zadatak. Neka $(X_1,X_2): \mathcal{N}(m_1, m_2, \sigma_1^2, \sigma_2^2, \rho)$. Ako je $\rho=0$, onda su X_1 i X_2 nezavisne.

rešenje: $\rho = 0$

Treba pokazati $\varphi_{(X_1,X_2)}(x_1,x_2) = \varphi_{X_1}(x_1)\varphi_{X_2}(x_2) \quad \forall x_1,x_2$

$$\varphi_{(X_1,X_2)}(x_1,x_2) \overset{\rho=0}{=} \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left[\left(\frac{x_1-m_1}{\sigma_1}\right)^2 + \left(\frac{x_2-m_2}{\sigma_2}\right)^2\right]}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{1}{2}\frac{(x_1-m_1)^2}{\sigma_1^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{1}{2}\frac{(x_2-m_2)^2}{\sigma_2^2}} = \varphi_{X_1}(x_1)\cdot\varphi_{X_2}(x_2)$$

$X: \mathcal{N}(m,\sigma^2)$

$$\varphi_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

Važi: X_1 i X_2 su nezavisne akko $\rho = 0$