

KOLOKVIJUM IZ UVODA U ANALIZU (M3 i M4 SMER)
4.12.2015.

1. Neka je $X = \mathbb{R} \setminus D_f$ gde je D_f domen funkcije

$$f(x) = \sqrt{\frac{2x^3 + 9x^2 - 6x - 5}{x^2 - 6x - 40}} \cdot \frac{1}{\sin \frac{1}{x}}.$$

- a) Odrediti infimum i supremum skupa X i proveriti da li skup X ima minimalni i maksimalni element.
- b) Odrediti unutrašnje, adherentne, rubne, izolovane i tačke nagomilavanja skupa X .

Rešenje:

Prvi korak je određivanje domena funkcije f :

- 1) Potkorena veličina mora biti nenegativna. Označimo sa

$$P(x) = 2x^3 + 9x^2 - 6x - 5, \quad Q(x) = x^2 - 6x - 40$$

Da bismo nasli nule polinoma $P(x)$ koristimo Hornerovu šemu:

$$\begin{array}{c|ccccc|c} & 2 & 9 & -6 & -5 \\ \hline 1 & 2 & 11 & 5 & 0 \end{array}$$

Dakle

$$P(x) = 2x^3 + 9x^2 - 6x - 5 = (x-1)(2x^2 + 11x + 5) = (x-1)(2x+1)(x+5)$$

$$Q(x) = x^2 - 6x - 40 = (x+4)(x-10)$$

Sada imamo

$$\begin{array}{c|ccccccc|c} & -5 & -4 & -\frac{1}{2} & 1 & 10 \\ \hline x-1 & - & - & - & - & + & + & + \\ \hline x+\frac{1}{2} & - & - & - & + & + & + & + \\ \hline x+5 & - & + & + & + & + & + & + \\ \hline x+4 & - & - & + & + & + & + & + \\ \hline x-10 & - & - & - & - & - & + & + \\ \hline P/Q & - & + & - & + & - & + & + \end{array}$$

2) Imamo jos uslove:

$$\begin{aligned} x &\neq 0, \quad \sin \frac{1}{x} \neq 0 \\ \Leftrightarrow x &\neq 0 \wedge \frac{1}{x} \neq \{k\pi, k \in \mathbb{Z}\} \\ \Leftrightarrow x &\neq 0 \wedge x \neq \{\frac{1}{k\pi}, k \in \mathbb{Z} \setminus \{0\}\} \end{aligned}$$

Dakle

$$D_f = [-5, -4) \cup [-\frac{1}{2}, 1] \cup (10, \infty) \setminus (\{\frac{1}{k\pi}, k \in \mathbb{Z} \setminus \{0\}\} \cup \{0\})$$

a onda je

$$X = (-\infty, -5) \cup [-4, -\frac{1}{2}) \cup (1, 10] \cup \{0\} \cup \{\frac{1}{k\pi}, k \in \mathbb{Z} \setminus \{0\}\}$$

Sada mozemo odrediti infimum, supremum i trazene skupove tacaka.

$$\inf X = -\infty, \quad \max X = 10 = \sup X$$

$$X^0 = (-\infty, -5) \cup (-4, -\frac{1}{2}) \cup (1, 10)$$

$$\overline{X} = (-\infty, -5] \cup [-4, -\frac{1}{2}] \cup [1, 10] \cup \{0\} \cup \{\frac{1}{k\pi}, k \in \mathbb{Z} \setminus \{0\}\}$$

$$X' = (-\infty, -5] \cup [-4, -\frac{1}{2}] \cup [1, 10] \cup \{0\}$$

$$\partial X = \{-5, -4, -\frac{1}{2}, 0, 1, 10\} \cup \{\frac{1}{k\pi}, k \in \mathbb{Z} \setminus \{0\}\}$$

$$iz X = \{\frac{1}{k\pi}, k \in \mathbb{Z} \setminus \{0\}\}$$

2. Dokazati da za svaki prirodan broj n važi

$$\sum_{k=1}^n \frac{k}{3^k} = \frac{3(3^n - 1) - 2n}{4 \cdot 3^n}.$$

Resenje:

Dokazujemo matematickom indukcijom:

BI $n = 1$:

$$\frac{1}{3} = \frac{3(3 - 1) - 2}{4 \cdot 3} = \frac{1}{3}$$

IH Prepostavimo da tvrdjenje vazi za neko $n \in \mathbb{N}$ tj.

$$\sum_{k=1}^n \frac{k}{3^k} = \frac{3(3^n - 1) - 2n}{4 \cdot 3^n}.$$

IK Pokazimo da tvrdjenje vazi za $n + 1$ tj.

$$\sum_{k=1}^{n+1} \frac{k}{3^k} = \frac{3(3^{n+1} - 1) - 2(n + 1)}{4 \cdot 3^{n+1}}.$$

$$\sum_{k=1}^{n+1} \frac{k}{3^k} = \sum_{k=1}^n \frac{k}{3^k} + \frac{n+1}{3^{n+1}} \stackrel{\text{IH}}{=} \frac{3(3^n - 1) - 2n}{4 \cdot 3^n} + \frac{n+1}{3^{n+1}} = \frac{3 \cdot 3(3^n - 1) - 2 \cdot 3n + 4(n+1)}{4 \cdot 3^{n+1}}$$

$$= \frac{3 \cdot 3^{n+1} - 2n - 5}{4 \cdot 3^{n+1}} = \frac{3(3^{n+1} - 1 - 2 - 2n)}{4 \cdot 3^{n+1}} = \frac{3(3^{n+1} - 1) - 2(n+1)}{4 \cdot 3^{n+1}}$$

sto je i trebalo pokazati.

3. (i) Dokazati po definiciji da je

$$\lim_{n \rightarrow \infty} \sqrt{4 + \frac{1}{n}} = 2.$$

Resenje

Neka je $\epsilon > 0$ dato. Trazimo n_0 tako da je za sve $n \geq n_0$

$$\begin{aligned} & \left| \sqrt{4 + \frac{1}{n}} - 2 \right| < \epsilon \\ & \Leftrightarrow \sqrt{4 + \frac{1}{n}} - 2 < \epsilon \\ & \sqrt{4 + \frac{1}{n}} - 2 = \sqrt{4 + \frac{1}{n}} - 2 \cdot \frac{\sqrt{4 + \frac{1}{n}} + 2}{\sqrt{4 + \frac{1}{n}} + 2} = \frac{\frac{1}{n}}{\sqrt{4 + \frac{1}{n}} + 2} < \frac{1}{4n} < \epsilon \\ & \Rightarrow n > \frac{1}{4\epsilon} \end{aligned}$$

pa za n_0 mozemo uzeti $n_0 = \lfloor \frac{1}{4\epsilon} \rfloor + 1$

2. nacin:

$$\sqrt{4 + \frac{1}{n}} - 2 < \epsilon \Leftrightarrow \sqrt{4 + \frac{1}{n}} < \epsilon + 2$$

$$4 + \frac{1}{n} < (\epsilon + 2)^2 \Leftrightarrow \frac{1}{n} < \epsilon^2 + 4\epsilon \Leftrightarrow n > \frac{1}{\epsilon^2 + 4\epsilon}$$

pa za n_0 mozemo uzeti $n_0 = \lfloor \frac{1}{\epsilon^2 + 4\epsilon} \rfloor + 1$

(ii) Odrediti sledeće granične vrednosti:

$$a) \lim_{n \rightarrow \infty} (\sqrt[3]{1 - n^3} + n)n^2; \quad b) \lim_{n \rightarrow \infty} \left(\frac{2015^n + 2}{2015^n} \right)^{2015^n}.$$

Resenje

a)

$$\begin{aligned} \lim_{n \rightarrow \infty} (\sqrt[3]{1 - n^3} + n)n^2 &= \lim_{n \rightarrow \infty} \frac{(1 - n^3 + n^3)n^2}{\sqrt[3]{(1 - n^3)^2} - n\sqrt[3]{1 - n^3} + n^2} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt[3]{\frac{1}{n^6} - \frac{2}{n^3} + 1} - \sqrt[3]{\frac{1}{n^3} - 1} + 1} = \frac{1}{3} \end{aligned}$$

b)

$$\lim_{n \rightarrow \infty} \left(\frac{2015^n + 2}{2015^n} \right)^{2015^n} = \lim_{n \rightarrow \infty} \left(1 + \frac{2}{2015^n} \right)^{2015^n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{\frac{2015^n}{2}} \right)^{\frac{2015^n}{2} \cdot 2} = e^2$$