Continuity properties of some classes of Wick and anti-Wick operators

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In this lecture we will present some results from recent contributions

N. Teofanov, J. Toft, *Pseudo-differential calculus in a Bargmann setting*, Annales Academiae Scientiarum Fennicae Mathematica, **45**, 227–257 (2020)

N. Teofanov, J. Toft, P. Wahlberg, *Pseudo-differential operators with isotropic symbols, and Wick and anti-Wick operators*, https://arxiv.org/abs/2011.00313

- part I

  background and motivation

- part II

  the Bargmann transform and
  analytic pseudo-differential operators,
  a sample of results

part I

- Hermite functions are given by

$$h_n(t) = (-1)^n \pi^{-1/4}(2^n n!)^{-1/2} e^{t^2/2}(e^{-t^2})^{(n)}$$
$$= e^{t^2/2} H_n(t), \quad t \in \mathbb{R}, \quad n = 0, 1, \ldots,$$

  and H_n are (normalized) Hermite polynomials.
- H_n were defined by Laplace in 1810, and later studied by Chebyshev (1859) and Hermite (1864).[1]
- N. Wiener used Hermite function expansions to prove the Plancherel formula for the Fourier transform around 1930.[2]
- In 1956. V. A. Steklov proved that the set of Hermite polynomials is dense in $L^2_{e^{-x^2}}(\mathbb{R})$.[3]

[1] Poincaré je bio Hermite-ov student, a Picard zet. Mihailo Petrović im je bio student, a zatim i prijatelj.

[2] N. Wiener, *Fourier integral and certain of its applications*, Cambridge University Press, London (1933)

[3] The result was probably known earlier.

- "Hermite functions are also of importance in quantum mechanics, as they are the wave functions for the stationary states of the quantum harmonic oscillator."[4]
- In 1970. B. Simon used Hermite function expansions in the framework of $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$.
- In 1986. S. Pilipović gave a characterization of Gelfand-Shilov type spaces through the growth condition of Hermite expansion coefficients.
- More recent related contributions are

M. Langenbruch, *Hermite functions and weighted spaces of generalized functions*, Manuscripta Math., 119, 269–285 (2006)

Z. Lozanov–Crvenkovic, D. Perisic, M. Taskovic, *Hermite expansions of elements of Gelfand-Shilov spaces in quasianalytic and non quasianalytic case,* Novi Sad J. Math., 37 (2), 129–147 (2007)

[4] G. B. Folland, Fourier Analysis and Its Applications, page 189,AMS, Providence, Rhode Island (1992)

Bollettino U. M. I.
(7) **2**-B (1988), 235-251

Tempered Ultradistributions.

STEVAN PILIPOVIC (*)

Sunto. – *Seguendo l'approccio di Komatsu alla teoria delle ultradistribuzioni di Beurling, si definisce e si studia una classe di ultradistribuzioni temperate* Σ'_α, $\alpha > \frac{1}{2}$, *che corrisponde alla classe di successioni* $p^{\alpha p}$, $\alpha > \frac{1}{2}$. Σ'_α *è invariante rispetto alla trasformazione di Fourier e i dati teoremi strutturati mostrano che essa è un'estensione naturale di* S'. *Si danno inoltre delle proprietà della convoluzione e della trasformazione di Fourier in* Σ'_α.

0. – Introduction.

Following the approach of Komatsu [8] to the theory of Beurling ultradistributions we define and investigate a class of tempered ultradistributions Σ'_α, $\alpha > \frac{1}{2}$, which correspond to the class of sequences $p^{\alpha p}$, $\alpha > \frac{1}{2}$. For $\alpha > 1$, the testing function space Σ_α contains $\mathcal{D}(p^{\alpha p})$ (in [8] $\mathcal{D}_{p^{\alpha p}}(\boldsymbol{R})$) as a dense subspace. For $\frac{1}{2} < \alpha < 1$, Σ_α is a subspace of the space of entire functions: it is « sufficiently rich » in the sense of Gel'fand and Shilov [4]. The space Σ_α, $\alpha > \frac{1}{2}$, is a dense subspace of $\mathcal{E}(p^{\alpha p})$ (in [8] $\mathcal{E}_{p^{\alpha p}}(\boldsymbol{R})$) and of the Schwartz space $\mathcal{S}$. Spaces Σ_α, $\alpha > \frac{1}{2}$, make a class of projective $S(m)$-spaces. General theory of such spaces can be found in Wloka [14]. More precisely, $\Sigma_\alpha \equiv S^{\alpha,0}_{\alpha,0}$, $\alpha > \frac{1}{2}$, where $S^{\alpha,0}_{\alpha,0}$, $\alpha > \frac{1}{2}$, are Gel'fand spaces (see [14, p. 89]). This class of Gel'fand spaces has not been studied in details up to now. Let us note that the various subspaces of the space of rapidly decreasing smooth functions were studied by Avantaggiati, Kashpirovskij and Cimmino.

We determine Σ_α by several equivalent sequences of norms. This enables us to show that Σ_α is invariant under the Fourier transformation and to investigate structural properties od this space. We give two representation theorems for elements of Σ'_α.

(*) This material is based on work supported by the U.S.-Yugoslav Joint Fund for Scientific and Technological Cooperation, in cooperation with the NSF under grant (JFP) 544.

- As an ingredient in estimates of Hermite coefficients in Gefand-Shilov type spaces, Pilipović used connections between the Hermite functions and the Hermite operator

$$\mathcal{R} = -\frac{d^2}{dx^2} + x^2.$$

- What is so special about the Hermite operator?
- $\mathcal{R}$ is one-dimensional Schrödinger operator with potential $V(x) = x^2$.
- $\mathcal{R} = \frac{1}{2}(AA^\dagger + A^\dagger A)$ where

$$A = -\frac{d}{dx} + x, \quad \text{and} \quad A^\dagger = \frac{d}{dx} + x$$

are called creation and annihilation operators since

$$Ah_k(x) = (2k+1)^{1/2} h_{k+1}(x), \quad \text{and} \quad A^\dagger h_k(x) = (2k)^{1/2} h_{k-1}(x).$$

- Hermite functions are eigenfunctions of the Hermite operator:

$$\mathcal{R} h_n = (2n+1) h_n.$$

- The commutator of (non-commuting) operators A, B is

$$[A, B] = AB - BA.$$

- For example, x and d/dx are non-commuting operators and from the Leibniz rule it follows that

$$\frac{d}{dx}(x\varphi) = \varphi + x\frac{d}{dx}\varphi \quad \Rightarrow \quad (\frac{d}{dx}x - x\frac{d}{dx})\varphi = \varphi,$$

so their commutator is the identity operator.
- "The Leibniz rule is essential for applications in mathematical physics."[5] For example, Hörmander's wavefront set is used to define the product of distributions which satisfies the Leibniz rule.
- In 1928. Fock described the use of creation annihilation operators to solve the commutator equation $[A, A^\dagger] = \mathrm{I}$.[6]
- Indeed, Fock identified A with the operator of multiplication with $z = x + iy$, and $A^\dagger$ with $\partial_z = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$.

[5] C. Brouder, N. V. Dang, F. Hélein, *A smooth introduction to the wavefront set* J. Phys. A: Math. Theor 47 (2014) 443001

[6] Famous Fock spaces are introduced in 1932.

- How the growth conditions on coefficients $a_n = (\varphi, h_n)$ in Hermite expansions are transferred to the condition on iterates/powers of $\mathcal{R}$?

$$
\begin{aligned}
|a_n|(2n+1)^{r+1} &= |(\varphi, h_n)|(2n+1)^{r+1} = |(\varphi, (2n+1)^{r+1} h_n)| \\
&= |(\varphi, \mathcal{R}^{r+1} h_n)| = |(\mathcal{R}^{r+1}\varphi, h_n)|.
\end{aligned}
$$

$$
\Longrightarrow \quad \sum_{n\in\mathbb{N}} |a_n|^2 (2n+1)^{2r+2} = \sum_{n\in\mathbb{N}} |(\mathcal{R}^{r+1}\varphi, h_n)|^2 = \|\mathcal{R}^{r+1}\varphi\|_{L^2}^2 .
$$

Then

$$
\varphi \in \mathcal{S}_\alpha^\alpha \quad \Leftrightarrow \quad \|\mathcal{R}^r \varphi\|_{L^2} \leqslant C h^r r!^{2\alpha}, \quad \alpha \geqslant \frac{1}{2},
$$

for some $h > 0$.[7] Notice:

$$
(\exists h > 0) \ \left(\sum_{n\in\mathbb{N}} |a_n|^2 (2n+1)^{2r} \right)^{1/2} \leqslant C h^r r!^{2\alpha}
$$

$$
\Longleftrightarrow \quad (\exists t > 0) \ |a_n| \leqslant C e^{-t(2n+1)^{1/2\alpha}} .
$$

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- In such a way isotropic Gelfand-Shilov spaces can be characterized (among other ways) both via the growth conditions of coefficients in Hermite expansions and the iterates of an operator.
- The $d-$dimensional Hermite operator $\mathcal{R} = -\Delta + |x|^2$ is a globally elliptic Shubin operator.
- Extensions to other operators are given by [Calvo, Rodino], [Cappiello, Gramchev, Rodino], [Pilipović, Prangoski, Vindas], [Vučković, Vindas].
- Description of other spaces in terms of the powers of harmonic oscillator are given by Toft, see also

A. Abdeljawad, C. Fernandez, A. Galbis, J. Toft, R. Üster, *Characterizations of a class of Pilipović spaces by powers of harmonic oscillator*, RACSAM 114, 131 (2020).

- Pilipović spaces (of Roumieu type) $\mathcal{H}_s(\mathbf{R}^d)$, $s \geqslant 0$, are given through the formal Hermite series expansions

$$f = \sum_{\alpha \in \mathbf{N}^d} c_\alpha h_\alpha, \quad c_\alpha = (f, h_\alpha), \quad |c_\alpha| \lesssim e^{-r|\alpha|^{1/2s}},$$

  some $r > 0$.
- Note that $\mathcal{H}_s(\mathbf{R}^d) \neq \mathcal{S}_s(\mathbf{R}^d) = \{0\}$, $1/2 > s \geqslant 0$.
- It was proved by Toft (2017) that

$$\mathcal{H}_s(\mathbf{R}^d) = \{f \mid \|\mathcal{R}^N f\|_{L^\infty} \lesssim h^N N!^{2s} \text{ for some } h > 0\}.$$

- Furthermore, Toft considered Pilipović *flat* spaces where the condition

$$|c_\alpha| \lesssim r^{|\alpha|} \alpha!^{-\frac{1}{2\sigma}}, \quad \sigma > 0$$

  some $r > 0$, is considered instead.

part II

- The Bargmann transform $\mathfrak{V}_d f$ of $f \in \mathcal{S}'_{1/2}(\mathbf{R}^d)$ is the entire function

$$\mathfrak{V}_d f(z) = \int_{\mathbf{R}^d} \mathfrak{A}_d(z,y) f(y)\, dy$$
$$= \pi^{-\frac{d}{4}} \int_{\mathbf{R}^d} \exp\Big(-\frac{1}{2}(\langle z,z\rangle + |y|^2) + 2^{1/2}\langle z,y\rangle\Big) f(y)\, dy,$$

$z \in \mathbf{C}^d$, and $\langle z,w\rangle = \sum_{j=1}^d z_j w_j$.

- It was proved by Bargmann in 1961. that

$$\mathfrak{V}_d : L^2(\mathbf{R}^d) \to A^2(\mathbf{C}^d)$$

is bijective and isometric from $L^2(\mathbf{R}^d)$ to the Fock space $A^2(\mathbf{C}^d)$, of entire functions with scalar product

$$(F,G)_{A^2} \equiv \int_{\mathbf{C}^d} F(z)\overline{G(z)}\, d\mu(z), \quad F, G \in A^2(\mathbf{C}^d),$$

where $d\mu(z) = \pi^{-d} e^{-|z|^2}\, d\lambda(z)$ ($d\lambda(z)$ is the Lebesgue measure on $\mathbf{C}^d$).

- In 1960's V. Bargmann[8] put a solid theoretical background for Fock's observations by showing that the Bargmann transform maps the creation and annihilation operators into multiplication and differentiation in the complex domain.
- By these investigations it follows that if

$$b(x,\xi) = \sum_{|\alpha+\beta|\leqslant N} c_1(\alpha,\beta)x^\alpha \xi^\beta,$$

then there is a unique

$$a(z,w) = \sum_{|\alpha+\beta|\leqslant N} c_2(\alpha,\beta)z^\alpha \overline{w}^\beta$$

such that $\mathrm{Op}_{\mathfrak{V}}(a) = \mathfrak{V}_d \circ \mathrm{Op}(b) \circ \mathfrak{V}_d^{-1}$ where the pseudodifferential operator $\mathrm{Op}(b)$ is given via the Kohn-Nirneberg correspondence

$$f(x) \mapsto (\mathrm{Op}(b)f)(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} b(x,\xi)\widehat{f}(\xi)e^{i\langle x,\xi\rangle}\, d\xi.$$

[8] Valentine Bargmann (1908–1989)

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. XIV, 187–214 (1961)

K. O. Friedrichs anniversary issue

On a Hilbert Space of Analytic Functions and an Associated Integral Transform

Part I

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1. Introduction

(a) The states of a quantum mechanical system of n degrees of freedom are usually described by functions either in configuration space (real variables $q_1, \cdots, q_n$) or in momentum space (real variables $p_1, \cdots, p_n$). Even in classical mechanics the complex combinations

$$\eta_k = 2^{-1/2}(q_k - ip_k), \qquad \xi_k = 2^{-1/2}(q_k + ip_k) \tag{1}$$

have proved useful. In quantum theory, these combinations are familiar from the treatment of the harmonic oscillator, and in addition they appear as creation and annihilation operators of Bose particles in field theory.

If q_k, p_k are selfadjoint operators satisfying the canonical commutation rules

$$[q_k, p_l] = i\delta_{kl}, \quad [q_k, q_l] = 0, \quad [p_k, p_l] = 0$$

(with Planck's constant $h = 2\pi$), then it follows that

$$\xi_k = \eta_k^*, \qquad \eta_k = \xi_k^*, \tag{2}$$

$$[\xi_k, \eta_l] = \delta_{kl}, \quad [\xi_k, \xi_l] = 0, \quad [\eta_k, \eta_l] = 0. \tag{3}$$

As early as 1928,[1] Fock introduced the operator solution $\xi_k = \partial/\partial\eta_k$ of the commutation rule $[\xi_k, \eta_k] = 1$, in analogy to Schrödinger's solution $p_k = -i\partial/\partial q_k$ of the relation $[q_k, p_k] = i$, and applied it to quantum field theory.

(b) It is the purpose of the present paper to study in greater detail the function space $\mathfrak{F}_n$ on which Fock's solution is realized, and its connection with the conventional Hilbert space $\mathfrak{H}_n$ of square integrable functions $\psi(q)$.

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*On leave from Princeton University.

[1] Fock, V., *Verallgemeinerung und Lösung der Diracschen statistischen Gleichung*. Z. Physik, Vol. 49, 1928, pp. 339–357. Focks' method has been further developed by Dirac. See, Dirac, P. A. M., *La seconde quantification*, Ann. Inst. H. Poincaré, Vol. 11, 1949, pp. 15–47.

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COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL XX, 1–101 (1967)

On a Hilbert Space of Analytic Functions and an Associated Integral Transform

Part II. A Family of Related Function Spaces Application to Distribution Theory

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Introduction

0.1. The problems treated in this paper arose in connection with the attempt to apply the methods of Part I[1] to the theory of tempered distributions.

Part I dealt with two Hilbert spaces, $\mathfrak{H}$ $(= \mathfrak{H}_n)$ and $\mathfrak{F}(= \mathfrak{F}_n)$, and the unitary mapping $\mathbf{A}$ of $\mathfrak{H}$ onto $\mathfrak{F}$. (In the following we omit the subscript n as long as the dimension n is fixed.) $\mathfrak{H}$ is the familiar Hilbert space $L^2(\mathbb{R}_n)$ of square integrable functions $\psi(q)$, where $q = (q_1, \cdots, q_n) \in \mathbb{R}_n$, based on the inner product

$$(\psi_1, \psi_2)_{\mathfrak{H}} = \int_{\mathbb{R}_n} \overline{\psi_1(q)}\psi_2(q)\, d^n q\,, \qquad d^n q = dq_1 \cdots dq_n\,.$$

$\mathfrak{F}$ may be described as follows. Let $\mathfrak{Z}$ $(= \mathfrak{Z}_n)$ be the set of all holomorphic (= entire analytic) functions $f(z)$ in n complex variables $(z = (z_1, \cdots, z_n),\ z_j = x_j + iy_j)$. The elements of $\mathfrak{F}$ are functions $f \in \mathfrak{Z}$, and the inner product in $\mathfrak{F}$ is

$$(g, f)_{\mathfrak{F}} = \int_{\mathbb{C}_n} \overline{g(z)} f(z)\, d\mu_n(z)\,, \qquad d\mu_n(z) = \pi^{-n} e^{-|z|^2} d^n z\,; \tag{0.1}$$

$\mathbb{C}_n$ is the n-dimensional complex Euclidean space, $|z|^2 = \sum_j |z_j|^2$, and $d^n z = \prod_{j=1}^{n} dx_j\, dy_j$.[2] In particular, $\|f\|^2 = (f, f) < \infty$ if $f \in \mathfrak{F}$.

[1] See [2], hereafter quoted as I or Part I. In the introduction to Part I it was said that Part II would deal with harmonic polynomials in $\mathfrak{F}$, and Part III with the rotation group. The projected Part III has appeared separately ([3]), and some results on harmonic polynomials are included in Section 8 of the present paper.

[2] Unless the domain of integration is explicitly indicated, all integrals extend over the whole range of the integration variables, i.e., $\mathbb{C}_n$ for z and $\mathbb{R}_n$ for q.

- The Bargmann transform maps the Hermite functions to monomials as

$$\mathfrak{V}_d h_\alpha = e_\alpha, \qquad e_\alpha(z) = \frac{z^\alpha}{\alpha!^{\frac{1}{2}}}, \quad z \in \mathbf{C}^d, \quad \alpha \in \mathbf{N}^d.$$

- The orthonormal basis $\{h_\alpha\}_{\alpha\in\mathbf{N}^d} \subseteq L^2(\mathbf{R}^d)$ is thus mapped to the orthonormal basis $\{e_\alpha\}_{\alpha\in\mathbf{N}^d} \subseteq A^2(\mathbf{C}^d)$.
- Let $\mathcal{A}_0(\mathbf{C}^d)$ be the set of all analytic polynomials of the form $F(z) = c(\alpha)e_\alpha(z)$, and let

$$\mathcal{A}_s(\mathbf{C}^d) = \{F(z) = c(\alpha)e_\alpha(z) \mid |c(\alpha)| \lesssim e^{-r|\alpha|^{1/2s}}\}, \quad s > 0.$$

  Then $\mathfrak{V}_d : \mathcal{H}_s(\mathbf{R}^d) \to \mathcal{A}_s(\mathbf{C}^d)$ is bijective mapping.
- We will also use $\widehat{\mathcal{A}}_s(\mathbf{C}^{2d}) = \{\, K\,;\, (z,w) \mapsto K(z,\overline{w}) \in \mathcal{A}_s(\mathbf{C}^{2d})\,\}$, $s \geqslant 0$.

- The starting point for our investigations are fundamental results given in

J. Toft, *The Bargmann transform on modulation and Gelfand-Shilov spaces, with applications to Toeplitz and pseudo-differential operators*, J. Pseudo-Differ. Oper. Appl. **3** (2012), 145–227.

J. Toft, *Images of function and distribution spaces under the Bargmann transform*, J. Pseudo-Differ. Oper. Appl. **8** (2017), 83–139.

- Let a be a locally bounded function on $\mathbf{C}^{2d}$ such that $(z,w) \mapsto a(z,w)$ is analytic, $z, w \in \mathbf{C}^d$. The Wick operator $\mathrm{Op}_{\mathfrak{W}}(a)$ with the symbol a is given by

$$\mathrm{Op}_{\mathfrak{W}}(a)F(z) = \pi^{-d} \int_{\mathbf{C}^d} a(z,w)F(w)e^{(z-w,w)}\, d\lambda(w),$$

where F is an entire function, $d\lambda$ is the Lebesgue measure and $(\,\cdot\,,\,\cdot\,)$ is the scalar product on $\mathbf{C}^d$.

- $(\mathrm{Op}_{\mathfrak{W}}(a)F)(z)$ is equal to the integral operator

$$(T_K F)(z) = \pi^{-d} \int_{\mathbf{C}^d} K(z,w)F(w)\, e^{-|w|^2}\, d\lambda(w) = \int_{\mathbf{C}^d} K(z,w)F(w)\, d\mu(w),$$

when $K(z,w) = K_a(z,w) = a(z,w)e^{(z,w)}$.

- Notice that

$$(\mathrm{Op}_{\mathfrak{W}}(z_j)F)(z) = z_j F(z) \quad \text{and} \quad (\mathrm{Op}_{\mathfrak{W}}(\overline{w}_j)F)(z) = (\partial_j F)(z)$$

when $F \in L^1((1+|w|)\, d\mu(w)) \cap A(\mathbf{C}^d)$.

Theorem

Let $s \geqslant \frac{1}{2}$. Then the following is true:

1. If T is a linear and continuous map from $\mathcal{A}_s'(\mathbf{C}^d)$ to $\mathcal{A}_s(\mathbf{C}^d)$, then there is a unique $a \in \widehat{A}(\mathbf{C}^d \times \mathbf{C}^d)$ such that

$$|a(z,w)| \lesssim e^{\frac{1}{2}\cdot|z-w|^2 - r(|z|^{\frac{1}{s}} + |w|^{\frac{1}{s}})}, \quad z, w \in \mathbf{C}^d,$$

   for some $r > 0$ and $T = \operatorname{Op}_{\mathfrak{V}}(a)$;

2. If T is a linear and continuous map from $\mathcal{A}_s(\mathbf{C}^d)$ to $\mathcal{A}_s'(\mathbf{C}^d)$, then there is a unique $a \in \widehat{A}(\mathbf{C}^d \times \mathbf{C}^d)$ such that

$$|a(z,w)| \lesssim e^{\frac{1}{2}\cdot|z-w|^2 + r(|z|^{\frac{1}{s}} + |w|^{\frac{1}{s}})}, \quad z, w \in \mathbf{C}^d,$$

   for every $r > 0$ and $T = \operatorname{Op}_{\mathfrak{V}}(a)$.

Theorem

Let $s \geqslant \frac{1}{2}$. Then the following is true:

1. If $a \in \widehat{A}(\mathbf{C}^d \times \mathbf{C}^d)$ satisfies

$$|a(z,w)| \lesssim e^{\frac{1}{2}\cdot|z-w|^2 - r(|z|^{\frac{1}{s}} + |w|^{\frac{1}{s}})}, \quad z,w \in \mathbf{C}^d,$$

   for some $r > 0$, then $\operatorname{Op}_{\mathfrak{V}}(a)$ from $\mathcal{A}_0(\mathbf{C}^d)$ to $\mathcal{A}_0'(\mathbf{C}^d)$ is uniquely extendable to a linear and continuous map from $\mathcal{A}_s'(\mathbf{C}^d)$ to $\mathcal{A}_s(\mathbf{C}^d)$;

2. If $a \in \widehat{A}(\mathbf{C}^d \times \mathbf{C}^d)$ satisfies

$$|a(z,w)| \lesssim e^{\frac{1}{2}\cdot|z-w|^2 + r(|z|^{\frac{1}{s}} + |w|^{\frac{1}{s}})}, \quad z,w \in \mathbf{C}^d,$$

   for every $r > 0$, then $\operatorname{Op}_{\mathfrak{V}}(a)$ from $\mathcal{A}_0(\mathbf{C}^d)$ to $\mathcal{A}_0'(\mathbf{C}^d)$ is uniquely extendable to a linear and continuous map from $\mathcal{A}_s(\mathbf{C}^d)$ to $\mathcal{A}_s'(\mathbf{C}^d)$.

- If $f, \phi \in \mathscr{S}(\mathbf{R}^d)$, then the short-time Fourier transform is defined by

$$V_\phi f(x,\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} f(y)\overline{\phi(y-x)}e^{-i\langle y,\xi\rangle}\,dy$$
$$= \mathscr{F}(f\,\overline{\phi(\,\cdot\,-x)})(\xi), \quad x,\xi \in \mathbf{R}^d.$$

- Let $\phi(x) = \pi^{-d/4}e^{-|x|^2/2}$, $x \in \mathbf{R}^d$. Then

$$\mathfrak{V}_d = U_{\mathfrak{V}} \circ V_\phi, \quad \text{and} \quad U_{\mathfrak{V}}^{-1} \circ \mathfrak{V}_d = V_\phi,$$

where

$$(U_{\mathfrak{V}}F)(x+i\xi) = (2\pi)^{d/2}e^{(|x|^2+|\xi|^2)/2}e^{-i\langle x,\xi\rangle}F(2^{1/2}x, -2^{1/2}\xi), \quad x,\xi \in \mathbf{R}^d.$$

- Let $\phi(x,\xi) = \pi^{-\frac{d}{2}}e^{i\langle x,\xi\rangle}e^{-\frac{1}{2}(|x|^2+|\xi|^2)}$, $x,\xi \in \mathbf{R}^d$, $a \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$ and let K_a be the kernel of $\operatorname{Op}(a)$.

$$e^{-\frac{1}{2}(|z|^2+|w|^2)}\mathfrak{V}_{\Theta,d}K_a(z,w)$$
$$= 2^{\frac{d}{2}}e^{-i(\langle x,\xi-2\eta\rangle+\langle y,\eta\rangle)}(V_\phi a)(\sqrt{2}x, -\sqrt{2}\eta, \sqrt{2}(\eta-\xi), \sqrt{2}(y-x))$$

when $z = x+i\xi \in \mathbf{C}^d$ and $w = y+i\eta \in \mathbf{C}^d$.

- We deduce continuity properties of operators when acting between suitable Lebesgue spaces of analytic functions.
- We recover some known results, and obtain new insights, since the conditions on weight functions are relaxed when the analytic pseudodifferential operators approach is used.

- Anti-Wick operators are Wick operators such that the symbol $a(z,w)$ does not depend on z:

$$\operatorname{Op}_{\mathfrak{V}}^{\mathrm{aw}}(a_0)F(z) = \pi^{-d}\int_{\mathbf{C}^d} a_0(w)F(w)e^{(z-w,w)}\,d\lambda(w).$$

- By Taylor expansion and integration by parts we get formally

$$\operatorname{Op}_{\mathfrak{V}}(a_0) = \sum_{\alpha\in\mathbf{N}^d}\frac{(-1)^{|\alpha|}}{\alpha!}\operatorname{Op}_{\mathfrak{V}}^{\mathrm{aw}}(b_\alpha), \qquad b_\alpha(w) = \partial_z^\alpha\overline{\partial}_w^\alpha a_0(w,w)$$

(provided a_0 fulfils some further conditions).

- The Bargmann assignment $S_{\mathfrak{V}}a$ of $a\in\mathcal{S}'_{1/2}(\mathbf{R}^{2d})$ is the unique element $a_0\in\widehat{A}(\mathbf{C}^{2d})$ which fulfills

$$\operatorname{Op}_{\mathfrak{V}}(a_0) = \mathfrak{V}_d\circ\operatorname{Op}^w(a)\circ\mathfrak{V}_d^{-1} \quad\Leftrightarrow\quad a_0 = S_{\mathfrak{V}}a.$$

- Let $0 \leqslant \rho \leqslant 1$, and let $\mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{R}^d)$ be the set of all $\omega \in \mathscr{P}(\mathbf{R}^d) \cap C^\infty(\mathbf{R}^d)$ such that

$$|\partial^\alpha \omega(x)| \lesssim \omega(x)\langle x\rangle^{-\rho|\alpha|}, \quad \alpha \in \mathbf{N}^d, \ \ x \in \mathbf{R}^d.$$

  Here $\mathscr{P}(\mathbf{R}^d)$ consists of weights that are $v-$moderate for a polynomially bounded weight.

- For $\omega \in \mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{R}^d)$ the Shubin symbol class $\mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^d)$ is the set of all $a \in C^\infty(\mathbf{R}^d)$ such that

$$|\partial^\alpha a(x)| \lesssim \omega(x)\langle x\rangle^{-\rho|\alpha|}, \qquad x \in \mathbf{R}^d,$$

  for every multi-index $\alpha \in \mathbf{N}^d$.

- We may now characterize the Shubin classes $\mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^d)$ by estimates on their Bargmann (kernel) assignments.

Theorem

Let $0 \leqslant \rho \leqslant 1$, $\omega \in \mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{R}^{2d})$ and $a \in \mathscr{S}'(\mathbf{R}^{2d})$. The following conditions are equivalent:

1. $a \in \mathrm{Sh}_{\rho}^{(\omega)}(\mathbf{R}^{2d})$,
2. $$\left|(\partial_z + \overline{\partial}_w)^{\alpha}(\partial_z - \overline{\partial}_w)^{\beta} S_{\mathfrak{V}} a(z,w)\right| \lesssim e^{\frac{1}{2}|z-w|^2}\omega(\sqrt{2}\,\overline{z})\langle z+w\rangle^{-\rho|\alpha+\beta|}\langle z-w\rangle^{-N},$$
   holds true for every $N \geqslant 0$ and $z, w \in \mathbf{C}^d$,
3. $$\left|\partial_z^{\alpha}\overline{\partial}_w^{\beta} S_{\mathfrak{V}} a(z,w)\right| \lesssim e^{\frac{1}{2}|z-w|^2}\omega(\sqrt{2}\,\overline{z})\langle z+w\rangle^{-\rho|\alpha+\beta|}\langle z-w\rangle^{-N},$$
   holds true for every $N \geqslant 0$ and $z, w \in \mathbf{C}^d$,

- By using our method we may recover the composition result (including the new result when $\rho = 0$).
- Recall, the product $a\#b$ of two symbols $a, b \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$ is defined as the product of symbols corresponding to operator composition:

$$\mathrm{Op}(a\#b) = \mathrm{Op}(a) \circ \mathrm{Op}(b).$$

- Let $0 \leqslant \rho \leqslant 1$ and $\omega_j \in \mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{R}^{2d})$ for $j = 1, 2$. If $a_j \in \mathrm{Sh}_\rho^{(\omega_j)}(\mathbf{R}^{2d})$ for $j = 1, 2$, then $a_1\#a_2 \in \mathrm{Sh}_\rho^{(\omega_1\omega_2)}(\mathbf{R}^{2d})$.
- We also deduce asymptotic expansion of the form

$$\mathrm{Op}_{\mathfrak{V}}(a) \sim \sum_{\alpha \in \mathbf{N}^d} \frac{(-1)^{|\alpha|}}{\alpha!} \mathrm{Op}_{\mathfrak{V}}^{\mathrm{aw}}(a_\alpha).$$

Theorem

Suppose $s > \frac{1}{2}$, $a \in \widehat{\mathcal{A}}'_s(\mathbf{C}^{2d})$, let $N \geqslant 1$ be an integer, and let

$$a_\alpha(w) = \partial_z^\alpha \overline{\partial}_w^\alpha a(w,w), \quad \alpha \in \mathbf{N}^d,$$

and

$$b_\alpha(z,w) = |\alpha| \int_0^1 (1-t)^{|\alpha|-1} \partial_z^\alpha \overline{\partial}_w^\alpha a(w + t(z-w), w)\, dt, \quad \alpha \in \mathbf{N}^d \backslash 0.$$

Then

$$\operatorname{Op}_{\mathfrak{V}}(a) = \sum_{|\alpha| \leqslant N} \frac{(-1)^{|\alpha|} \operatorname{Op}_{\mathfrak{V}}^{\mathrm{aw}}(a_\alpha)}{\alpha!} + \sum_{|\alpha| = N+1} \frac{(-1)^{|\alpha|} \operatorname{Op}_{\mathfrak{V}}(b_\alpha)}{\alpha!}.$$

- Let $0 \leqslant \rho \leqslant 1$ and $\omega \in \mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{R}^{2d})$. Then $S_{\mathfrak{V}}$ is a homeomorphism from $\mathrm{Sh}_{\rho}^{(\omega)}(\mathbf{R}^{2d})$ to $\widehat{\mathcal{A}}_{\mathrm{Sh},\rho}^{(\omega)}(\mathbf{C}^{2d})$,

$$\left|\partial_z^{\alpha}\overline{\partial}_w^{\beta}a_0(z,w)\right| \leqslant Ce^{\frac{1}{2}|z-w|^2}\omega(\sqrt{2}\,\overline{z})\langle z+w\rangle^{-\rho|\alpha+\beta|}\langle z-w\rangle^{-N}, \quad N \geqslant 0.$$

- We have the following version of the sharp Garding inequality.[9]

[9]The real counterpart represents one of the basic applications of the Anti-Wick theory, cf. Nicola, Rodino, Global Pseudo-differential calculus on Euclidean spaces, page 59, Birkhäuser Verlag, Basel, 2010.

Theorem

Let $\rho > 0$, $\omega(z) = \langle z \rangle^{2\rho}$ and let $a_0 \in \widehat{\mathcal{A}}^{(\omega)}_{\mathrm{Sh},\rho}(\mathbf{C}^{2d})$ be such that $a_0(w, w) \geqslant -C_0$ for all $w \in \mathbf{C}^d$, for some constant $C_0 \geqslant 0$. Then

$$\mathrm{Re}((\mathrm{Op}_{\mathfrak{V}}(a_0)F, F)_{A^2}) \geqslant -C\|F\|^2_{A^2}, \qquad F \in \mathcal{A}_{\mathscr{S}}(\mathbf{C}^d)$$

and

$$|\,\mathrm{Im}((\mathrm{Op}_{\mathfrak{V}}(a_0)F, F)_{A^2})| \leqslant C\|F\|^2_{A^2}, \qquad F \in \mathcal{A}_{\mathscr{S}}(\mathbf{C}^d)$$

for some constant $C \geqslant 0$.

for your kind attention!