Beyond Gevrey regularity

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This lecture is dedicated to the memory of professor Todor Gramchev (June 13, 1956 – October 18, 2015)

Contents of the lecture:

- introduction - historical remarks,
- Gevrey classes in different contexts,
- beyond Gevrey classes,
- microlocal analysis for extended Gevrey regularity...

- Let $U \in \mathbb{R}$ be an open set and let $f \in C^\infty(U)$. f is real analytic on U ($f \in \mathcal{A}(U)$) if and only if $\forall x \in U$ there exists an open interval V, $x \in V \subseteq U$, $\exists A > 0$, $\exists C > 0$ s. t.

$$|f^{(n)}(x)| \leq CA^n n!, \quad n = 0, 1, 2, \ldots, \forall x \in V.$$

- If $f \in \mathcal{A}(U)$ and $x_0 \in U$, then the uniqueness property holds:

$$|f^{(n)}(x_0)| = 0, \; n = 0, 1, 2, \cdots \Rightarrow f \equiv 0.$$

- Borel's theorem (1895): For a given sequence $\{c_n\}$ of real numbers and $x_0 \in \mathbb{R}$, there exists $f \in C^\infty$ in a neighborhood of x_0 such that

$$f^{(n)}(x_0) = c_n, \quad n = 0, 1, 2, \ldots$$

- The Hadamard “uniqueness“ problem (1912): Describe *quasianalytic classes* via sequences (M_n):
  Let $C(M_n) \subset C^\infty(\mathbb{R})$ such that

$$|f^{(n)}(x)| \leq CA^n M_n, \quad n = 0, 1, 2, \ldots, x \in \mathbb{R}.$$

  $C(M_n)$ are known as *Carleman classes*. Cf.

Mandelbrojt, S.: Analytic functions and classes of infinitely differentiable functions. Rice Inst. Pamphlet 29 (1), (1942)

- $C(M_n)$ is quasianalytic class if the uniqueness property holds:

$$|f^{(n)}(x_0)| = 0, \; n = 0, 1, 2, \cdots \Rightarrow f \equiv 0.$$

- $\mathcal{A}(\mathbb{R}) = C(n!)$, and and $f \in C(n!)$ can be extended to bounded holomorphic functions in a strip $|\Im(z)| < \delta$ for some $\delta > 0$.

- Let $M_0 = 1$ and $M_n^2 \leq M_{n-1}M_{n+1}$. The following are equivalent (Carleman (1926), Ostrowski (1930)):
  a) $C(M_n)$ is quasi-analytic.
  b) $\sum \left(\frac{1}{M_n}\right)^{1/n} = \infty$.
  c) $\sum \frac{M_{n-1}}{M_n} = \infty$.
- $M_n = n!(\ln n)^n(\ln(\ln n))^n \Rightarrow$ quasianalyticity. ($M_n = n!^s$, $s \in (0, 1]$)
- $M_n = n!^s$, $s > 1$, $\Rightarrow$ non– quasianalyticity.
- $C(M_n)$ is a quasi-analytic class iff it does not contain compactly supported functions (apart from 0).
- Localization procedures using smooth ”cut off” functions apply to non-quasianalytic classes.

When $d > 1$ the usual notion/notation extension is used.

- Analytic class, $\mathcal{A}(\mathbb{R}^d) : (\exists C > 0) \ \left|\partial^\beta f(x)\right| \le C^{1+|\beta|}\beta!, \quad \beta \in \mathbb{N}_0^d.$
- Gevrey class, $G^s(\mathbb{R}^d), s \ge 1$:

$$(\exists C > 0)\left|\partial^\beta f(x)\right| \le C^{1+|\beta|}\beta!^s, \ \ \beta \in \mathbb{N}_0^d.$$

  If Ω is an open subset in $\mathbb{R}^d$ then the above estimate holds for all $x \in K$, $K \Subset \Omega$, and $C = C_K$. Compactly supported Gevrey functions are denoted by $G_0^s(\Omega) = G^s(\Omega) \cap C_0^\infty(\Omega)$.
- Gelfand-Shilov space, $\mathcal{S}_\mu^\nu(\mathbb{R}^d)$, $\nu, \mu \ge 0$: $(\exists C > 0)$ $(\exists \varepsilon > 0)$

$$\left|\partial^\beta f(x)\right| \le C^{1+|\beta|}\beta!^\nu e^{-\varepsilon|x|^{1/\mu}}, \ \ \forall x \in \mathbb{R}^d, \ \ \forall \beta \in \mathbb{N}_0^d.$$

Hadamard: $|\partial^\beta f(x)| \le C^{1+|\beta|}M_{|\beta|}, \quad x \in \mathbb{R}^d, \ \beta \in \mathbb{N}_0^d.$
Gelfand-Shilov: $|x^\alpha\partial^\beta f(x)| \le C^{1+|\alpha|+|\beta|}M_{|\alpha|,|\beta|}, \quad x \in \mathbb{R}^d, \ \alpha, \beta \in \mathbb{N}_0^d.$

"Whenever the properties of certain differential operators differ in the C^∞ and in the analytic category, it is natural to investigate the behavior of such operators in the scale of Gevrey classes G^s and, if possible, to find the critical value(s) of s, i.e. those for which a change of behavior occurs."[1]

- ✓ Holmgren (1901): If the solution of $\partial v/\partial y = \partial^2 v/\partial x^2$ is analytic with respect to x, then $v = v(x, y)$ is in $G^2(\mathbb{R})$ in y variable.
- The heat equation is the prototype of the equations with hyperbolic principal symbol for which the Cauchy problem admits solutions if data belong to $G^s(\mathbb{R})$, for sufficiently small $s > 1$.
- The non-strictly hyperbolic Cauchy problem is C^∞ well posed if additional (Levi) conditions are assumed, but G^s well posed with a restriction on $s > 1$ but without any assumptions of lower order terms.
- For linear partial differential operator with constant coefficients, if all the roots of the characteristic equation are real, then the Cauchy problem is well posed in G^s for $1 \leq s < m/(m+1)$, where m is the multiplicity of the roots.

[1] Todor Gramchev, in Encyclopaedia of Mathematics, Supplement Vol. II, Springer, 2000.

- ✓ The attractor for the laser equations is contained in every Gevrey Class:
    - Menon, Govind: Gevrey class regularity for the attractor of the laser equations. Nonlinearity 12 (1999), no. 6, 1505-1510.
- ✓ A non-analytic solution of PDE which belongs to a class contained in every Gevrey Class:
    - Hörmander, Lars: A counterexample of Gevrey class to the uniqueness of the Cauchy problem, Math. Research Letters 7 (2000) 615–624
- ✓ Gevrey hypoellipticity of sums of squares of vector fields/Treves conjecture
    - Paolo Albano, Paolo; Bove, Antonio: Wave Front Set of Solutions to Sums of Squares of Vector Fields, Memoirs AMS, 2013.
    - Paolo Albano, Paolo; Bove, Antonio; Mughetti Marco: Analytic Hypoellipticity for sums of squares and the Treves conjecture, Arxiv, 12 May 2016.

- ✓ Time varying operators ($\int h(t, t-s)x(s)ds$) with symbols (time-varying impulse responses) in Gevrey Classes:

Volevic, L. R.: Pseudodifferential operators with holomorphic symbols, and Gevrey classes, Trudy Moskov. Mat. Obc. 24 (1971), 43-68.

- ✓ Gevrey-type pseudodifferential operators:

Boutet de Monvel, Louis; Kre, Paul: Pseudo-differential operators and Gevrey classes, Ann. Inst. Fourier 17 (1) (1967), 295-323.

Hashimoto, S.; Morimoto, Y.; Matsuzawa, T.: Opérateurs pseudodifférentiels et classes de Gevrey, Comm. Partial Differential Equations 8 (12) (1983), 1277-1289.

Zanghirati, Luisa: Pseudodifferential operators of infinite order and Gevrey classes. Ann. Univ. Ferrara Sez. VII (N.S.) 31 (1985), 197-219.

- ✓ Micro-local analysis in the context of Gevrey classes

Rodino, Luigi: Linear partial differential operators in Gevrey spaces. World Scientific, 1993.

- If $f \in G_0^s(\mathbb{R}^d)$ then $\exists C, \varepsilon > 0$

$$\left|\hat{f}(\xi)\right| \leq Ce^{-\varepsilon|\xi|^{1/s}}, \quad \xi \in \mathbb{R}^d. \tag{1}$$

- Let $(G^s(\mathbb{R}^d))' = \mathcal{E}_s'(\mathbb{R}^d)$, $s > 1$. If $f \in \mathcal{E}_s'(\mathbb{R}^d)$ (or $f \in \mathcal{S}'(\mathbb{R}^d)$) and (1) holds then $f \in G^s(\mathbb{R}^d)$.

✓ There exists a wavelet ψ associated to a MRA such that $\psi \in C^\infty \setminus \cup_{s>1} G^s$ and $\hat{\psi} \in \cap_{s>1} G^s$, see

Moritoh, Shinya; Tomoeda, Kyoko: A further decay estimate for the Dziubanski-Hernndez wavelets. Canad. Math. Bull. 53 (1) (2010), 133-139.

Fukuda, Naohiro; Kinoshita, Tamotu; Uehara, Ion On the wavelets having Gevrey regularities and subexponential decays. Math. Nachr. 287 (2014), no. 5-6, 546-560.

"I have nothing against the invention of new function spaces, but – if possible – one should see a *chance* for possible usefulness."[2]

- ✓ Gevrey-Sobolev spaces:
  - Chen, Hua; Rodino, Luigi: General theory of PDE and Gevrey classes. 1996.
- ✓ Gevrey modulation spaces:
  - Iwabuchi, Tsukasa: Navier-Stokes equations and nonlinear heat equations in modulation spaces with negative derivative indices. J. Differential Equations 248 (8) (2010), 1972-2002.
  - Wang, Baoxiang; Huo, Zhaohui; Hao, Chengchun; Guo, Zihua: Harmonic analysis method for nonlinear evolution equations. I. 2011.
  - Cordero, Elena; Nicola, Fabio; Rodino, Luigi: Gabor representations of evolution operators. Trans. Amer. Math. Soc. 367 (2015), no. 11, 7639-7663.
  - Reich, Maximilian; Reissig, Michael; Sickel, Winfried: Non-analytic superposition results on modulation spaces with subexponential weights. J. Pseudo-Differ. Oper. Appl., online first 2016.

[2] H. G. Feichtinger, Choosing function spaces in harmonic analysis, 2015.

- Let $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $p \in \mathbf{N}$, $\tau > 0$ and $\sigma > 1$ ($M_0 = 1$). Then $(M_p^{\tau,\sigma})^2 \leq M_{p-1}^{\tau,\sigma} M_{p+1}^{\tau,\sigma}$ and $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$ satisfies the non-quasianalyticity condition.
- Let $\tau > 0$, $\sigma > 1$, $h > 0$, and $K \subset\subset U$, where U is an open set in $\mathbf{R}^d$. Then

$$\mathcal{E}_{\tau,\sigma,h}(K) = \{\phi \in C^\infty(U) \mid (\exists A > 0) |\partial^\alpha \phi(x)| \leq A h^{|\alpha|^\sigma} M_{|\alpha|}^{\tau,\sigma},\ \alpha \in \mathbf{N}^d, x \in K\}$$

is a Banach space with the norm given by the infimum of the constants A.

- We are interesting in inductive and projective limit spaces:

$$\mathcal{E}_{\{\tau,\sigma\}}(U) = \varprojlim_{K \subset\subset U} \varinjlim_{h \to \infty} \mathcal{E}_{\tau,\sigma,h}(K),$$

$$\mathcal{E}_{(\tau,\sigma)}(U) = \varprojlim_{K \subset\subset U} \varprojlim_{h \to 0} \mathcal{E}_{\tau,\sigma,h}(K).$$

- If $s > 1$ and $\sigma = 1$, then $\mathcal{E}_{\{s,1\}}(U) = G^s(U)$ are Gevrey classes, and $\mathcal{E}_{\{1,1\}}(U) = \mathcal{A}(U)$.

- The corresponding Carleman class: $C(M_p^{\tau,\sigma})(U)$ is contained in $\mathcal{E}_{\tau,\sigma}(U)$, and we use abbreviated notation τ,σ for $\{\tau,\sigma\}$ or (τ,σ).
- If $0<\tau_1<\tau_2$, then $\mathcal{E}_{\{\tau_1,\sigma\}}(U)\hookrightarrow\mathcal{E}_{(\tau_2,\sigma)}(U)\hookrightarrow\mathcal{E}_{\{\tau_2,\sigma\}}(U)$, $\sigma>1$, and

$$\cup_{s>1}G^s(U)\hookrightarrow\mathcal{E}_{\tau,\sigma}(U),\quad \tau>0,\sigma>1.$$

- $\mathcal{E}_{\tau,\sigma}(U)$ are algebras closed under the action of $P(x,D)=\sum_{|\alpha|\leq m}a_\alpha(x)\partial^\alpha$, with $a_\alpha\in\mathcal{E}_{\tau,\sigma}(U)$, although $M_p^{\tau,\sigma}$ does not satisfy "stability under differentiation" condition.
  We also considered certain ultradifferentiable operators on $\mathcal{E}_{\tau,\sigma}(U)$.
- The space $\mathcal{E}_{\{\tau,\sigma\}}(U)$ is inverse closed in $C^\infty(U)$.
  This result is used in the construction of a function in $\mathcal{E}_{\tau,\sigma}(U)\setminus\cup_{s>1}G^s(U)$.

- Let $\tau > 0$, $\sigma > 1$, $\Omega \subseteq K \subset\subset U \subseteq \mathbf{R}^d$, where Ω and U are open in $\mathbf{R}^d$, and the closure of Ω is contained in K.
- Let $u \in \mathcal{D}'(U)$. We study regularity related to the condition

$$|\widehat{u}_N(\xi)| \leq A\,\frac{h^N N!^{\tau/\sigma}}{|\xi|^{\lfloor N^{1/\sigma}\rfloor}}, \quad N \in \mathbf{N},\ \xi \in \mathbf{R}^d \backslash \{0\}. \tag{2}$$

  where $\{u_N\}_{N\in\mathbf{N}}$ is bounded sequence in $\mathcal{E}'(U)$ such that $u_N = u$ in Ω and A, h are some positive constants.

We apply a change of variables which "speeds up" or "slows down" the decay estimates of single members of the corresponding sequences, while retaining the asymptotic behavior when $N \to \infty$, and call such procedure *enumeration*. For example,

$$|\widehat{u}_N(\xi)| \leq B\,\frac{k^N N!^{1/\sigma}}{|\xi|^{\lfloor (N/\tau)^{1/\sigma}\rfloor}},$$

for some $B, k > 0$ and for all $N \in \mathbf{N}$ and $\xi \in \mathbf{R}^d \backslash \{0\}$, is obtained from (2) after an enumeration.

Definition

Let $\tau > 0$ and $\sigma > 1$, $u \in \mathcal{D}'(U)$, and $(x_0, \xi_0) \in U \times \mathbf{R}^d \backslash \{0\}$. Then $(x_0, \xi_0) \notin \mathrm{WF}_{\{\tau,\sigma\}}(u)$ (resp. $\mathrm{WF}_{(\tau,\sigma)}(u)$) if there exists open neighborhood $\Omega \subset U$ of x_0, a conic neighborhood Γ of ξ_0, and a bounded sequence $\{u_N\}_{N\in\mathbf{N}}$ in $\mathcal{E}'(U)$ such that $u_N = u$ on Ω and (2) holds for some constants $A, h > 0$ (resp. for every $h > 0$ there exists $A > 0$).

Let $u \in \mathcal{D}'(U)$, $t > 1$. Then for $0 < \tau < \rho$ and $\sigma > 1$ it holds

$$\mathrm{WF}(u) \subseteq \mathrm{WF}_{\{\rho,\sigma\}}(u) \subseteq \mathrm{WF}_{(\rho,\sigma)}(u) \subseteq \mathrm{WF}_{\{\tau,\sigma\}}(u)$$

$$\subseteq \bigcap_{t>1} \mathrm{WF}_t(u) \subseteq \mathrm{WF}_A(u),$$

where WF_t and WF_A are Gevrey and analytic wave front sets, respectively.

Theorem

Let $u \in \mathcal{D}'(U)$ and let $P(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$ be partial differential operator of order m such that $a_\alpha(x) \in \mathcal{E}_{\{\tau,\sigma\}}(U)$, $|\alpha| \leq m$, $\tau > 0$, $\sigma > 1$. Then if $P(x,D)u = f$ in $\mathcal{D}'(U)$, it holds

$$\mathrm{WF}_{\{2^{2\sigma-1}\tau,\sigma\}}(f) \subseteq \mathrm{WF}_{\{2^{2\sigma-1}\tau,\sigma\}}(u) \subseteq \mathrm{WF}_{\{\tau,\sigma\}}(f) \cup \mathrm{Char}(P(x,D)).$$

In particular,

$$\mathrm{WF}_{0,\infty}(P(x,D)u) \subseteq \mathrm{WF}_{0,\infty}(u) \subseteq \mathrm{WF}_{0,\infty}(P(x,D)u) \cup \mathrm{Char}(P(x,D)),$$

where $\mathrm{WF}_{0,\infty}(u) = \bigcup_{\sigma>1} \bigcap_{\tau>0} \mathrm{WF}_{\{\tau,\sigma\}}(u)$.

Stevan Pilipovic, Nenad Teofanov, Filip Tomić: On a class of ultradifferentiable functions, NSJOM 45 (1) (2015), 125-142.

Stevan Pilipovic, Nenad Teofanov, Filip Tomić: Beyond Gevrey regularity, Journal of Pseudo-Differential Operators and Applications, 7 (2016), 113–140.

Nenad Teofanov, Filip Tomić: Ultradifferentiable functions of class $M_p^{t,s}$ and microlocal regularity, to appear in "Generalized functions and Fourier analysis", Birkhäuser, 2017.

Stevan Pilipovic, Nenad Teofanov, Filip Tomić: Wave front sets for extended Gevrey regularity, in preparation.