Report:
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Calculating friction velocity and heat flux in the surface layer using Monin-Obukhov similarity theory

This report focuses on the surface layer in the atmosphere. In the surface layer, which is a part of planetary boundary layer, heat flux and friction velocity can be considered constant with change of height. The purpose of this report is to calculate friction velocity and surface heat flux in different cases of atmospheric stability, and explain how wind sheer and buoyancy contribute to the production of turbulent kinetic energy. This calculation will be preformed using Monin-Obukhov similarity theory.

Introduction

In the paper of Obukhov [1], a new scaling parameter in the surface layer of the atmosphere has been introduced. This parameter had the dimension of length and therefore is known as *Obukhov length* (L). Later on, in the paper of Monin and Obukhov [2] a new theory for calculating wind profile and potential temperature profile in stable and unstable atmosphere was given. By using Buckingham Π-theorem two dimensionless groups have been formed, one for wind profile and the other for potential temperature profile (left side of equation (1) and (2)) in the surface layer. Because these groups can evaluate to a dimensionless number they can be treated as functions of some other relevant dimensionless group [5], and for that reason it was possible to write equations (1) and (2) in this form.

$$\frac{kz}{u_\star}\frac{du}{dz} = \Phi_m\left(\frac{z}{L}\right) \tag{1}$$

$$\frac{z}{-\frac{(\overline{w\theta})_s}{u_\star}}\frac{d\theta}{dz} = \Phi_h\left(\frac{z}{L}\right) \tag{2}$$

These functions, Φ_m and Φ_h, are different from one another, and both of them depend on the dimensionless group $\frac{z}{L} = \xi$ (see equation (3)) which turns out to be a very important scaling parameter in the surface layer. Depending on the sign of ξ it is possible to tell if the atmosphere is stable (positive) or unstable (negative). Which makes parameter ξ *stability sensitive.*

$$\xi = \frac{z}{L} = \frac{z}{-\frac{\overline{\theta_v}{u_\star}^3}{kg\left(\overline{w\theta}\right)_s}} \tag{3}$$

Fraction in the denominator of equation (3) represents Obukhov length, where $\overline{\theta_v}$ is mean potential temperature in the atmosphere, k is von Kármán constant, g is gravitational acceleration, $u_\star$ is friction velocity and $\left(\overline{w\theta}\right)_s$ is the surface heat flux. It is important to notice that equation (1) looks very similar to the equation (4), which is commonly used for calculating wind profile above a flat, homogeneous surface when the atmosphere is neutrally stratified.

$$\frac{du}{dz} = \frac{u_\star}{kz} \tag{4}$$

By saying that in case of neutral stratification $\Phi_m(\xi) = 1$, equation (1) would be exactly the same as equation (4). This means that the function $\Phi_m(\xi)$ is in fact a *universal function* which can be used to calculate the wind profile above a surface in every kind of atmospheric stability, and atmospheric stability is defined with ξ in the following way:

$$\xi = \begin{cases} < 0, \text{unstable stratification} \\ = 0, \text{neutral stratification} \\ > 0, \text{stable stratification} \end{cases}$$

Same applies for universal function $\Phi_h(\xi)$, it can be used for calculating potential temperature profile, or surface heat flux profile as we shall see.

Basic Equations

If we start from equation (1) and (2) then it is possible to get to equations (5) and (6), see appendix A.

$$u(z) = \frac{u_\star}{k}\left(\int_{z_0}^{z}\frac{dz}{z} - \int_{\xi_0}^{\xi}\frac{1-\Phi_m(\xi)}{\xi}d\xi\right) \tag{5}$$

$$\theta(z) - \theta(z_0) = -\frac{\left(\overline{w\theta}\right)_s}{ku_\star}\left(\int_{z_0}^{z}\frac{dz}{z} - \int_{\xi_0}^{\xi}\frac{1-\Phi_h(\xi)}{\xi}d\xi\right) \tag{6}$$

Solving these equations for $u_\star$ and $\left(\overline{w\theta}\right)_s$ we can calculate the values for friction velocity and surface heat flux, considering that we are familiar with the values of the left side of equations which indicate what is the wind velocity in equation (5) and the temperature difference between the higher and lower boundary of the surface layer, equation (6). Only problem is that the functions Φ_m and Φ_h have different analytic forms in case of unstable, neutral and stable atmosphere.

Unstable and neutral atmosphere

In the case of unstable and neutral atmosphere we will consider that the universal functions have the form given by equation (7) and (8), which was proposed in the paper [3], and with coefficient for wind profile $a_m = 11.5$ and for potential temperature profile $a_h = 12.2$.

$$\Phi_m = (1 + a_m\xi)^{-\frac{1}{3}} \tag{7}$$

$$\Phi_h = (0.74 + a_h\xi)^{-\frac{1}{3}} \tag{8}$$

Substituting these functions into equations (5) and (6), and than solving integrals, see appendix B, we can obtain equations for friction velocity $u_\star$ and for surface heat flux $\left(\overline{w\theta}\right)_s$ *in the case of unstable and neutral atmosphere*, as given bellow.

$$u_\star = \frac{u(z)k}{\log\frac{z}{z_{0m}} - \frac{3}{2}\log\frac{1+(1+a_m\frac{z}{L})^{\frac{1}{3}}+(1+a_m\frac{z}{L})^{\frac{2}{3}}}{1+(1+a_m\frac{z_{0m}}{L})^{\frac{1}{3}}+(1+a_m\frac{z_{0m}}{L})^{\frac{2}{3}}} + \sqrt{3}\left(\arctan\frac{2(1+a_m\frac{z}{L})^{\frac{1}{3}}+1}{\sqrt{3}} - \arctan\frac{2(1+a_m\frac{z_{0m}}{L})^{\frac{1}{3}}+1}{\sqrt{3}}\right)}$$

$$\left(\overline{w\theta}\right)_s = -\frac{u_\star k\left(\theta(z) - \theta(z_{0_h})\right)}{\log\frac{z}{z_{0_h}} - \frac{3}{2}\log\frac{1+(0.74+a_h\frac{z}{L})^{\frac{1}{3}}+(0.74+a_h\frac{z}{L})^{\frac{2}{3}}}{1+(0.74+a_h\frac{z_{0_h}}{L})^{\frac{1}{3}}+(0.74+a_h\frac{z_{0_h}}{L})^{\frac{2}{3}}} + \sqrt{3}\left(\arctan\frac{2(0.74+a_h\frac{z}{L})^{\frac{1}{3}}+1}{\sqrt{3}} - \arctan\frac{2(0.74+a_h\frac{z_{0_h}}{L})^{\frac{1}{3}}+1}{\sqrt{3}}\right)}$$

Stable atmosphere

When considering a stable atmosphere, universal functions have the form given by equation (9) and (10), as it was proposed in the paper [4].

$$\Phi_m = 1 + 4.7\xi \tag{9}$$

$$\Phi_h = 0.74(1 + 4.7\xi) \tag{10}$$

Procedure is similar to the previously explained. We substitute universal functions from equation (9) and (10) into equation (5) and (6) and obtain equations for friction velocity $u_\star$ and for surface heat flux $\left(\overline{w\theta}\right)_s$ *in the case of stable atmosphere*, equations (11) and (12).

$$u_\star = \frac{u(z)k}{\log\frac{z}{z_{0m}} - 4.7\frac{z - z_{0m}}{L}} \tag{11}$$

$$\left(\overline{w\theta}\right)_s = -\frac{u_\star k(\theta(z) - \theta(z_{0_h}))}{\log\frac{z}{z_{0_h}} - 4.7\frac{z - z_{0_h}}{L}} \tag{12}$$

Equations that are derived in this section allow us to calculate friction velocity and surface heat flux. These exact equations are implemented in the model made for the pupose of this paper, and they make the model work.

Results

In order to calculate the values for friction velocity $u_\star$ and surface heat flux $\left(\overline{w\theta}\right)_s$ *initial conditions* have been set up. Their purpose is to describe the physical situation in the atmosphere that produces $u_\star$ and $\left(\overline{w\theta}\right)_s$:

- *Upper boundary for the surface layer*, bellow this value ($z = 2\text{m}$) we consider that the fluxes are constant with the change of height.
- *Lower boundary for the surface layer*, this parameter depends on the type of surface. In our case ($z_{0_m} = 0.1\text{m}$, $z_{0_h} = 0.01\text{m}$) is a flat surface, above which we are calculating $u_\star$ and $\left(\overline{w\theta}\right)_s$.
- *Initial values for wind velocity* at the upper boundary of the surface layer ($u(z = 2\text{m})$).
- *Initial values for potential temperature difference*, difference between the upper and the lower boundary of a surface layer. This value indicates to the atmospheric stability ($\theta(z) - \theta(z_{0_h})$).

We have set up initial values for wind velocity and potential temperature difference so that we can see how friction velocity and surface heat flux change depending on the atmospheric stability.

$$u(z)\left[\mathrm{m\,s^{-1}}\right] = \begin{cases} 0.1,\ 0.2,\ 0.5,\ 1 & \text{calm wind} \\ 2,\ 5,\ 10,\ 20,\ 50 & \text{strong wind} \end{cases}$$

$$\theta(z) - \theta(z_{o_h})\,[\mathrm{K}] = \begin{cases} -1,\ -2,\ -3,\ \ldots,-8,\ -9,\ -10 & \text{unstable atmosphere} \\ 0 & \text{neutral atmosphere} \\ +1,\ +2,\ +3,\ \ldots,+8,\ +9,\ +10 & \text{stable atmosphere} \end{cases}$$

There are two situations that are covered in this paper. First is when we consider that the wind is *calm* ($u(z) < 2\,\mathrm{m\,s^{-1}}$) and the second when the wind is *strong* ($u(z) \geq 2\,\mathrm{m\,s^{-1}}$). Using previously derived equations, and setting up parameters and initial conditions in the way they are set in this paper, it is possible to compute the results and obtain Figure 1.

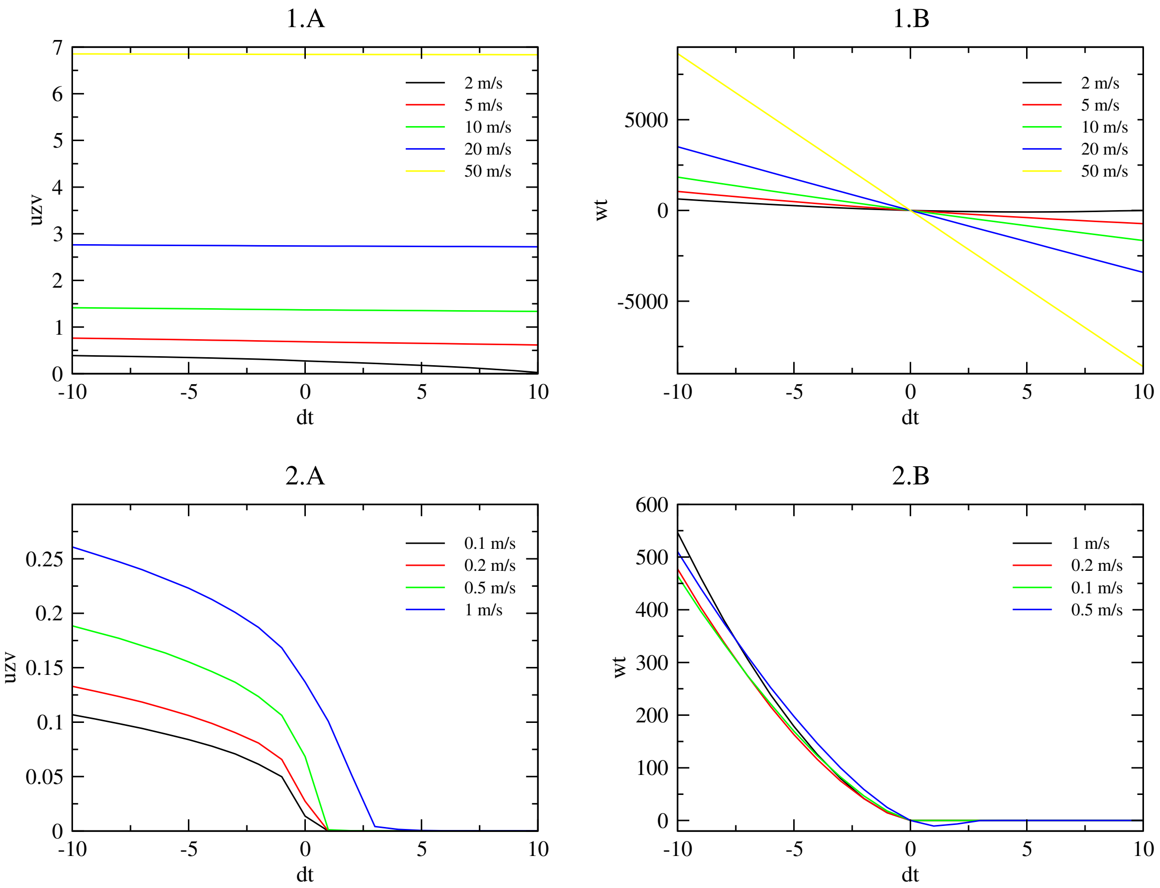

Figure 1: Upper plots represent friction velocity $[\mathrm{m\,s^{-1}}]$ (uzv) and surface heat flux $[\mathrm{W/m^2}]$ (wt) in case of strong winds, and lower plots represent values in case of calm winds, all in the function of potential temperature difference.

We can tell that in the case of calm winds friction velocity decreases as the stability in the atmosphere increases (2.A), and when the atmosphere is *stable enough* than friction velocity equals zero, and in that case surface heat flux equals zero as well (2.B). As for the stronger winds (1.A), they behave similar as calm winds, but they would need much more stable atmosphere in order for friction velocity to reach zero. It is worth noticing that in case of strong winds the surface heat flux takes both positive (upward flux from surface towards the atmosphere) and negative values (downward flux from atmosphere towards the surface), depending on the atmospheric stability.

Qualitative analysis for TKE

These results can be related with the change of *Turbulent Kinetic Energy* (TKE) in the surface layer given by equation (13)[1]. Value of TKE changes because of the three effects: wind sheer (SPROD) and buoyancy (BPROD), these effects can generate turbulence, and dissipation (DISS) which is always responsible for lowering the rate of change of TKE. We can relate our results with TKE since friction velocity is proportional to SPROD and surface heat flux is proportional to BPROD. Because we are not interested in dissipation for this analysis DISS can be neglected.

$$\frac{\partial}{\partial t}\mathrm{TKE} = \mathrm{SPROD} + \mathrm{BPROD} - \mathrm{DISS} \tag{13}$$

In Figure 1. we can see that friction velocity can take only positive values and we can conclude that SPROD will always increase TKE. As for the surface heat flux things are more complex because it can be either positive or negative (depending on the atmospheric stability), which means that BPROD can increase or decrease the amount of TKE. So in the case of unstable atmosphere it is easy to see that both SPROD and BPROD would contribute in increasing TKE. But in the case of stable atmosphere, TKE will increase only if $\mathrm{SPROD} > |\mathrm{BPROD}|$, or it will decrease if $\mathrm{SPROD} < |\mathrm{BPROD}|$.

[1] More information on TKE and the derivation of this equation can be found in [5].

Appendix A

If we start from equation (1), separating the variables and than integrating the whole equation from the roughness length for wind z_{0_m} to some height z, we would get to equation (A1).

$$\int_{z_{0_m}}^{z} du = \int_{z_{0_m}}^{z} \frac{u_\star}{k} \frac{\Phi_m\left(\frac{z}{L}\right)}{z} dz \tag{A1}$$

Because the boundaries of the integral are within the surface layer, friction velocity $u_\star$ behaves as a constant and goes in front of the integral. Now if we add and subtract 1 inside of the integral we can form two separate integrals, and that way get to equation (A2).

$$\int_{z_{0_m}}^{z} du = \frac{u_\star}{k} \left(\int_{z_{0_m}}^{z} \frac{dz}{z} - \int_{z_{0_m}}^{z} \frac{1-\Phi_m\left(\frac{z}{L}\right)}{z} dz \right) \tag{A2}$$

If we make a derivative of equation (3), and than substitute the result back into equation (A2), as well as solve the integral on the left side of the equation (A2), we would get to equation (A3).

$$u(z) - u(z_{0_m}) = \frac{u_\star}{k} \left(\int_{z_0}^{z} \frac{dz}{z} - \int_{\xi_0}^{\xi} \frac{1-\Phi_m(\xi)}{\xi} d\xi \right) \tag{A3}$$

Now if we recall on the basics of the theory about roughness length for wind, *it is the height at which wind velocity equals zero*. So making $u(z_{0_h}) = 0$ we would obtain equation (5). By going trough this procedure only starting from equation (2) it is possible to obtain equation (6).

Appendix B

By using the form for universal function given by equation (7) and starting from equation (5) it is possible to obtain equation for friction velocity $u_\star$ in unstable and neutral atmosphere. When we substitute equation (7) into equation (5) we would get to equation (B1).

$$u(z) = \frac{u_\star}{k} \left(\int_{z_0}^{z} \frac{dz}{z} - \int_{\xi_0}^{\xi} \frac{1-(1+a_m\xi)^{-\frac{1}{3}}}{\xi} d\xi \right) \tag{B1}$$

To make the procedure more convenient for the reader, we introduce J as given in equation (B2).

$$J = \int_{\xi_0}^{\xi} \frac{1-(1+a_m\xi)^{-\frac{1}{3}}}{\xi} d\xi \tag{B2}$$

In order to solve the integral in (B2) we could use a helpful substitution $x = (1+a_m\xi)^{\frac{1}{3}}$, that way $\xi = \frac{x^3-1}{a_m}$ and $d\xi = \frac{3x^2}{a_m}dx$. By substituting all of it back into equation (B2) we would obtain equation (B3).

$$J = 3\int_{x_0}^{x} \frac{x^2-x}{x^3-1} dx \tag{B3}$$

Now if we bring out x in the numerator $x^2 - x = x(x-1)$, and than use an alternative form of $x^3 - 1 = (x-1)(x^2+x+1)$, we would get to the following form of the integral with the solution given in equation (B4).

$$J = \frac{3}{2}\log\frac{1+x+x^2}{1+x_0+{x_0}^2} + \sqrt{3}\left(\arctan\frac{2x_0+1}{\sqrt{3}} - \arctan\frac{2x+1}{\sqrt{3}}\right) \tag{B4}$$

Substituting the solution for J back into equation (B1) and solving the rest of the equation we obtain equation (B5).

$$\begin{aligned} u(z) =& \frac{u_\star}{k}\log\frac{z}{z_{0_m}} - \frac{u_\star}{k}\frac{3}{2}\log\frac{1+x+x^2}{1+x_0+{x_0}^2} \\ &+ \frac{u_\star}{k}\sqrt{3}\left(\arctan\frac{2x+1}{\sqrt{3}} - \arctan\frac{2x_0+1}{\sqrt{3}}\right) \end{aligned} \tag{B5}$$

With some minor rearrangements we can express everything as a function of friction velocity $u_\star$ and substitute back ξ instead of x. Same procedure applies for getting to equation for surface heat flux $\left(\overline{w\theta}\right)_s$ in case of unstable and stable atmosphere.

References

[1] A.M.Obukhov. *Turbulence in an atmosphere with a non-uniform temperature*. 1948.

[2] A.M.Obukhov A.S.Monin. *Basic laws of turbulent mixing in the surface layer of the atmosphere*. 1954.

[3] G.Mellor. *Analytic prediction of the properties of stratified planetary surf ace layers*. 1973.

[4] Y.Izumi E.F.Bradley J.A.Businger, J.C.Wyngraad. *Flux-profile relationships in the atmosphereic surface layer*. 1973.

[5] R.B.Stull. *An Introduction to Boundary Layer Meteorology*. 1988.