Писмени испит из увода у анализу, октобарски рок 2021.

1. Одредити домен и асимптоте функције
$f(x) = \frac{x^3}{2(x+1)^2}$.

Решење. $(x+1)^2 \neq 0 \iff x \neq -1$

$\Rightarrow D_f = \mathbb{R} \setminus \{-1\} = (-\infty, -1) \cup (-1, +\infty)$.

В.А $\lim\limits_{x \to -1-} \frac{x^3}{2(x+1)^2} = \frac{-1}{0^+} = -\infty$

$\lim\limits_{x \to -1+} \frac{x^3}{2(x+1)^2} = \frac{-1}{0^+} = -\infty$

$\Rightarrow$ $x = -1$ је В.А кад $x \to -1\pm$

Х.А $\lim\limits_{x \to \pm\infty} \frac{x^3}{2(x+1)^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} =$

$\lim\limits_{x \to \pm\infty} \frac{x}{2\left(1 + \cancel{\frac{1}{x}}^{\,0}\right)^2} = \frac{\pm\infty}{2} = \pm\infty$

$\Rightarrow$ нема Х.А. кад $x \to \pm\infty$

К.А $k = \lim\limits_{x \to \pm\infty} \frac{f(x)}{x} = \lim\limits_{x \to \pm\infty} \frac{1}{2}\left(\frac{x}{x+1}\right)^2 = \frac{1}{2}$

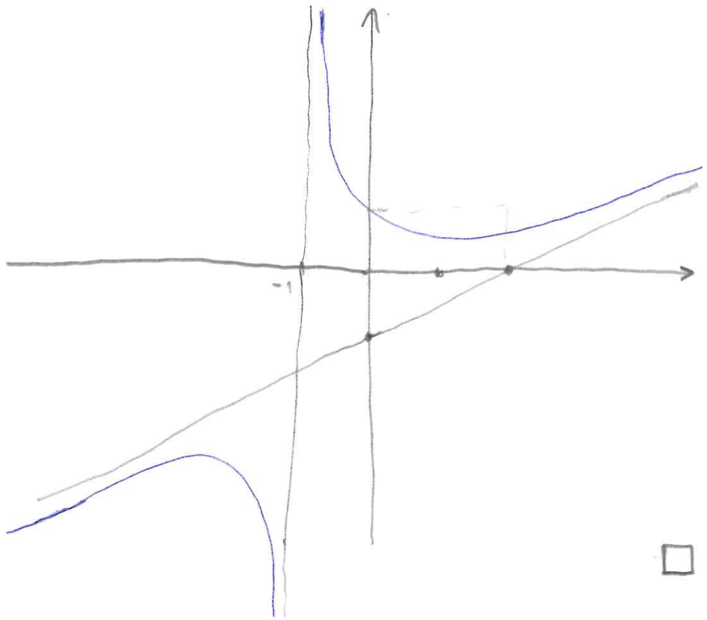
$n = \lim\limits_{x \to \pm\infty} \left(f(x) - \frac{1}{2}x\right) =$

$= \lim\limits_{x \to \pm\infty} \frac{x^3 - x(x+1)^2}{2(x+1)^2} =$

$= \lim\limits_{x \to \pm\infty} \frac{-2x^2 - x}{2(x^2 + 2x + 1)} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} =$

$= \lim\limits_{x \to \pm\infty} \frac{-2 - \frac{1}{x}}{2 + \frac{4}{x} + \frac{2}{x^2}} = \frac{-2}{2} = -1$

$\Rightarrow y = \frac{1}{2}x - 1$ је К.А. кад $x \to \pm\infty$.



□

2. Нека је $X = \{x \in \mathbb{R} : \frac{3x^2 + x + 12}{2x^2 + 7x + 3} \leq 1\}$.

(а) Наћи $\inf X$, $\sup X$, $\min X$, $\max X$,

(б) Наћи X°, $\overline{X}$, ∂X, X', X^{iz}.

Решење: $\frac{3x^2 + x + 12}{2x^2 + 7x + 3} \leq 1 \iff$

$\frac{3x^2 + x + 12 - 2x^2 - 7x - 3}{2x^2 + 7x + 3} \leq 0$

$\frac{x^2 - 6x + 9}{2x^2 + 7x + 3} \leq 0 \iff$

$\frac{(x-3)^2}{2x^2 + 7x + 3} \leq 0 \iff$

$x_{1,2} = \frac{-7 \pm \sqrt{49 - 4 \cdot 3 \cdot 2}}{4} = \frac{-7 \pm 5}{4}$ (-3, $-\frac{1}{2}$)

$\frac{(x-3)^2}{2(x+3)(x+\frac{1}{2})} \leq 0$

	$-\infty$ … -3	-3 … $-\frac{1}{2}$	$-\frac{1}{2}$ … 3	3	3 … $+\infty$
$(x-3)^2$	+	+	+	0	+
$(x+3)$	−	+	+		+
$x+\frac{1}{2}$	−	−	+		+
	+	−	+	0	+

$\Rightarrow X = (-3, -\frac{1}{2}) \cup \{3\}$.

(а) $\inf X = -3 \notin X \Rightarrow \min X$ не $\exists$

$\sup X = 3 \in X \Rightarrow \max X = 3$

(б) $X^\circ = (-3, -\frac{1}{2})$,

$\overline{X} = X$,

$\partial X = \{-3, -\frac{1}{2}, 3\}$,

$X' = X \setminus \{3\}$,

$X^{iz} = \{3\}$. □

3. (а) Одредити све тачке нагомилавања низа $f_n = \frac{n}{n+1}\sin\frac{3n\pi}{4}$, $n \in \mathbb{N}$, као и подизове који тим тачкама теже. Наћи $\liminf(f_n)$, $\limsup(f_n)$.

Решење:

$n=1 \rightarrow \sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$,

$n=2 \rightarrow \sin\frac{6\pi}{4} = -1$,

$n=3 \rightarrow \sin\frac{9\pi}{4} = \frac{\sqrt{2}}{2}$,

$n=4 \rightarrow \sin 3\pi = 0$

$n=5 \rightarrow \sin\frac{15\pi}{4} = \frac{\sqrt{2}}{2}$,

$n=6 \rightarrow \sin\frac{18\pi}{4} = -1$,

$n=7 \rightarrow \sin\frac{21\pi}{4} = \frac{\sqrt{2}}{2}$

$n=8 \rightarrow \sin 6\pi = 0$

Тачке нагомилавања су $\frac{\sqrt{2}}{2}, 0, -1$,

$\liminf(f_n) = -1$, $\limsup(f_n) = \frac{\sqrt{2}}{2}$,

$f_2, f_6, f_{10}, \ldots \longrightarrow -1$,

$f_1, f_3, f_5, f_7 \ldots \longrightarrow \frac{\sqrt{2}}{2}$. □

(б) Наћи $\lim\limits_{n\to\infty} \sqrt[n]{2^n+3^n+5^n}$.

Решење:
$$\lim_{n\to\infty}\sqrt[n]{2^n+3^n+5^n} =$$
$$\lim_{n\to\infty}\sqrt[n]{5^n\left(\frac{2^n}{5^n}+\frac{3^n}{5^n}+1\right)} =$$
$$5\cdot\lim_{n\to\infty}\Big(\underbrace{\left(\tfrac{2}{5}\right)^n}_{0}+\underbrace{\left(\tfrac{3}{5}\right)^n}_{0}+1\Big)^{\overbrace{\frac{1}{n}}^{0}} =$$
$$5\cdot \overbrace{1}^{=1} = 5.$$

Други начин:
$$5 = \sqrt[n]{5^n} \le \sqrt[n]{2^n+3^n+5^n} \le \sqrt[n]{3\cdot5^n} = 3^{\frac{1}{n}}\cdot 5$$
ТОУН

$\rightarrow 5$ □

④ (а) Израчунати
$$\lim_{x\to0}\left(\frac{1}{\mathrm{tg}\,x}-\frac{1}{\sin x}\right).$$

Решење:
$$\lim_{x\to0}\left(\frac{1}{\mathrm{tg}\,x}-\frac{1}{\sin x}\right) =$$
$$\lim_{x\to0}\left(\frac{\cos x}{\sin x}-\frac{1}{\sin x}\right) =$$
$$\lim_{x\to0}\Big(\underbrace{\frac{\cos x-1}{x^2}}_{-\frac{1}{2}}\cdot\underbrace{\frac{x}{\sin x}}_{1}\cdot x\Big) =$$
$$\left(-\frac{1}{2}\right)\cdot1\cdot0 = 0.$$ □

(б) $$\lim_{x\to3}\frac{\sqrt{4x-3}-3}{\sqrt{x+1}-2}.$$

Решење:
$$\lim_{x\to3}\frac{\sqrt{4x-3}-3}{\sqrt{x+1}-2}\cdot\frac{\sqrt{4x-3}+3}{+|-}\cdot\frac{\sqrt{x+1}+2}{+|-} =$$
$$\lim_{x\to3}\frac{4x-3-9}{x+1-4}\cdot\frac{\sqrt{x+1}+2}{\sqrt{4x-3}+3} =$$
$$\lim_{x\to3}\frac{4\cancel{(x-3)}}{\cancel{x-3}}\cdot\frac{\sqrt{x+1}+2}{\sqrt{4x-3}+3} = 4\cdot\frac{2+2}{3+3} = \frac{16}{6}$$
$$= \frac{8}{3}.$$ □

5. Дата је ф-ја $f: \mathbb{R} \to \mathbb{R}$ са

$$f(x) = \begin{cases} \dfrac{\sin(5\pi x)}{\sin(\pi x)}, & 0 < x < 1, \\ ax^2 - b, & x \in (-\infty, 0] \cup [1, +\infty). \end{cases}$$

Наћи непознате константе a и b (ако је могуће) тако да f буде непрекидна на $\mathbb{R}$.

Решење: Треба да важи

$$\lim_{x \to 0-} (ax^2 - b) = \lim_{x \to 0+} \frac{\sin(5\pi x)}{\sin(\pi x)},$$

$$\lim_{x \to 1-} \frac{\sin(5\pi x)}{\sin(\pi x)} = \lim_{x \to 1+} (ax^2 - b).$$

Важи $\displaystyle\lim_{x \to 0+} \frac{\sin(5\pi x)}{\sin(\pi x)} = \lim_{x \to 0} \underbrace{\frac{\sin(5\pi x)}{5\pi x}}_{1} \cdot \underbrace{\frac{5\pi x}{\sin(\pi x)}}_{5}$

$$= 5.$$

Дакле, $a \cdot 0^2 - b = 5 \Rightarrow \boxed{b = -5}$.

Такође, $\displaystyle\lim_{x \to 1-} \frac{\sin 5\pi x}{\sin(\pi x)} = \left\{ \begin{array}{c} \text{смена:} \\ x = t+1 \\ t \to 0- \end{array} \right\} =$

$$\lim_{t \to 0-} \frac{\sin(5\pi t + 5\pi)}{\sin(\pi t + \pi)} = \lim_{t \to 0-} \frac{\sin 5\pi t \underbrace{\cos 5\pi}_{-1} + \cos 5\pi t \underbrace{\sin 5\pi}_{0}}{\sin \pi t \underbrace{\cos \pi}_{=-1} + \cos \pi t \underbrace{\sin \pi}_{0}} =$$

$$\lim_{t \to 0} \frac{\sin 5\pi t}{\sin \pi t} = \lim_{t \to 0} \underbrace{\frac{\sin(5\pi t)}{5\pi t}}_{1} \cdot \underbrace{\frac{5\pi t}{\sin(\pi t)}}_{5} = 5,$$

$\Longrightarrow a \cdot 1^2 - b = 5,$

$a - (-5) = 5$

$a + 5 = 5 \Longrightarrow \boxed{a = 0}$

$\square$