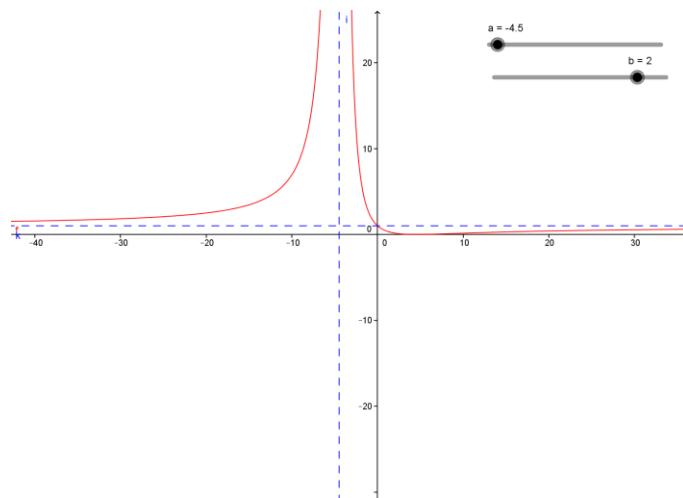


# The impact of dynamic properties of the software packages Mathematica and GeoGebra to the examining and graphing of functions with parameters

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# Introduction

Examining the functions and drawing their graphs is one of the most important teaching contents of Calculus or Mathematics I, which are studied during the first semester of bachelor studies of physics, chemistry, technical sciences, etc. This teaching content is also studied in the fourth grade of high school and the most of vocational schools. During examining and graphing the functions students should consider their essential characteristics. This is necessary for later application of examined functions. Earlier studies have indicated that using of appropriate software simplifies function examining and graphing to students and students who used GeoGebra application during this learning process were more successful than students who studied function examining without any software.

During function examining, students are supposed to determine its domain, zeros, sign, derivate, monotonicity, extreme points, inflection points and convexity and, finally, to draw its graph. In order to analyze of functions characteristics be as efficient as possible, it's important to consider influence of individual parameters on function characteristics. Analyzing of parameters influence on function characteristics without the use of adequate software requires a lot of time, because it is necessary to examine the function with as much as possible different values of observed parameter (more values provide a more reliable result). Besides required time, this kind of analysis presents difficulties to many students. Due to the complexity of the function examining procedure, many students make mistakes in some steps of this procedure. One example of significant mistake made by some students is mixing up the zeros and domain of the rational function (e.g. in function  $f(x) = \frac{x-3}{x-2}$ , domain is  $\mathbb{R} \setminus \{3\}$ , and zero is  $x = 2$ , which is wrong).

On the other side, using of dynamic software enables analyzing of parameters influence on function characteristics to be carried out very quickly. By using this kind of software it is possibly to change a large number of parameter values in a short time and to observe influence of changing parameter values on individual function characteristics, which can help students to better understand each characteristic and not to make mistakes as previously mentioned. This teaching material contains several examples of using software packages Mathematica and GeoGebra in function examining and graphing.

## The use of dynamic software in function examining

Using of dynamic software significantly reduces the time required for process of function examining, but also enables several functions, which belong to the same class of functions, to be examined simultaneously. Class of functions is determined by one parent function, which contains one or more parameters. Assignment of different values to those parameters, different functions, which belong to the same class, are being formed. We can also say that those functions are derived from the same parent function. Although they are different, functions which belong to the same class have some common characteristics. By function examining and graphing we identify which characteristic are common for these functions and which are different. Possibilities incurred by using of dynamic software in function examining and graphing should contribute to better understanding of influence of parameters on function characteristics by the students.

One of the tasks that we can ask students is to analyze parent function which corresponds to the class of quadratic functions. The form of this function would be:  $f(x) = ax^2 + bx + c$ . This parent function contains three parameters ( $a$ ,  $b$  and  $c$ ). In Mathematica and GeoGebra we present these parameters by the sliders. By moving one of three sliders we are changing the value of appropriate parameter. Students should do the above procedure and then they should determine the domain, zeros, sign, derivate, monotonicity, extreme points, inflection points and convexity and, finally, to draw graph for a several quadratic functions defined by assigning different values to parameters  $a$ ,  $b$  and  $c$ . Their most important task would be to analyze the influence of parameters on individual function characteristics. They should move the sliders and observe and notice changes of function characteristics. It would be expected from students to conclude which characteristic depends on which parameters and to determine kind of that dependence.

Using of dynamic software also allows incitement students to reverse thinking. One of the tasks that could be required from the students is to observe several functions with concrete values of parameters and to reach a conclusion, based on these functions characteristics, which of the observed functions belong to the same class and, also, to determine their parent function. This task can be required from the students only after they have overcame influence of the parameters on function characteristics.

Both of software packages, Mathematica and GeoGebra, allow work with sliders, so we can examine function with parameters by using any of these two packages. The most important

difference between Mathematica and GeoGebra is that GeoGebra is free software package and Mathematica is very expensive and also GeoGebra is much simpler for use, even for high school students. On the other side, Mathematica provides greater opportunities. Also, it is not necessary for students to be trained to use Mathematica and it is not necessary for students to have Mathematica installed on their computers, because teacher can prepare a few templates for function examining and save those templates as CDF files, which can be opened with CDF player, which is free. By using CDF player, students can observe function characteristics and graph, they also can change parameter values by moving the sliders and observe changing function characteristics, and that is everything we need to show to the students. The following examples show how we can use dynamic software as aid in function examining.

***Example 1:***

Examine and graph the next functions:  $f_1(x) = 2x^2 + 5x + 2$ ,  $f_2(x) = -2x^2 + 3x + 1$ ,  $f_3(x) = -x^2 - 3x + 2$ ,  $f_4(x) = 3x^2 + 4x$ ,  $f_5(x) = 1.5x^2 + 3x + 2$ .

***Solution:***

Examining five functions requires a lot of time. In this case, student can recognize that all of these functions belong to the class of quadratic functions, so they have many common characteristics. Because of that, we can use Mathematica or GeoGebra templates for examining the function  $f(x) = ax^2 + bx + c$ , and by setting appropriate values to the slider  $a$ ,  $b$  and  $c$ , we can examine all of these five functions in a very short time.

Of course, it is necessary for students to be familiar with procedure of function examining, so, if they didn't do the function examining and graphing before, we should show them at least one example without using software. Consequently, we are going to examine and graph the function  $f_1(x)$ , and then we are going to examine the remaining function by using Mathematica and GeoGebra applications.

*Characteristics of the function  $f_1(x) = 2x^2 + 5x + 2$ :*

Domain:

Function  $f_1(x)$  is defined on the set of real numbers ( $D = \mathbb{R}$ )

$$\text{Zeros: } x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad x_1 = -2 \quad x_2 = -0.5$$

Intersection with the y-axis:

$f_1(0) = 2$ , which implies that the graph of this function intersects the y-axis at the point  $(0, 2)$ .

The function  $f_1(x)$  doesn't have asymptotes.

Sign:  $f_1(x) > 0$  when  $x \in (-\infty, -2) \cup (-0.5, \infty)$

$f_1(x) < 0$  when  $x \in (-2, -0.5)$

The first derivate:  $f'_1(x) = 4x + 5$

Monotonicity:  $f'_1(x) = 0$  when  $x = -1.25$

$f'_1(x) > 0$  when  $x > -1.25 \Rightarrow$  Function increases

$f'_1(x) < 0$  when  $x < -1.25 \Rightarrow$  Function decreases

The second derivate:  $f''_1(x) = 5$

Extreme points:  $f_1(-1.25) = -1.125$

$f''_1(x) > 0 \Rightarrow$  the function has a minimum point at  $(-1.25, -1.125)$

The function  $f_1(x)$  is convex and it doesn't have inflection points.

Graph:

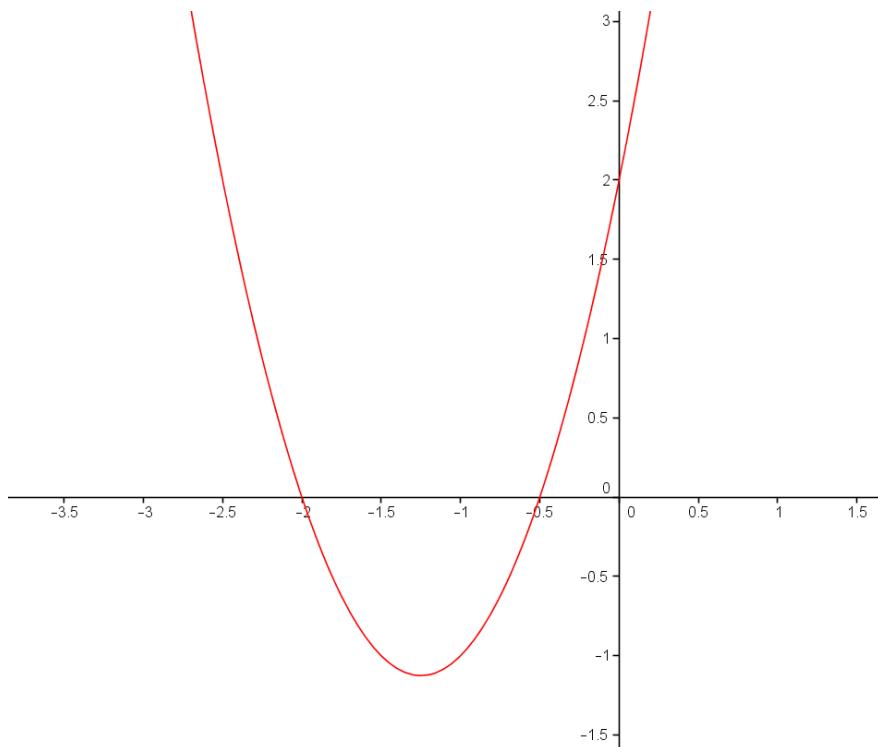


Figure 1

Remaining functions will be examined by using Mathematica or GeoGebra.

Click on [WM1](#) opens the window into the software package Wolfram Mathematica with interactive display of the function  $f(x) = ax^2 + bx + c$ .

Click on [GGB1](#) opens the window into the software package GeoGebra with interactive display of the function  $f(x) = ax^2 + bx + c$ .

The following figures show the graphs of the functions  $f_2(x)$  in Wolfram Mathematica work environment, or  $f_4(x)$  in GeoGebra work environment, respectively.

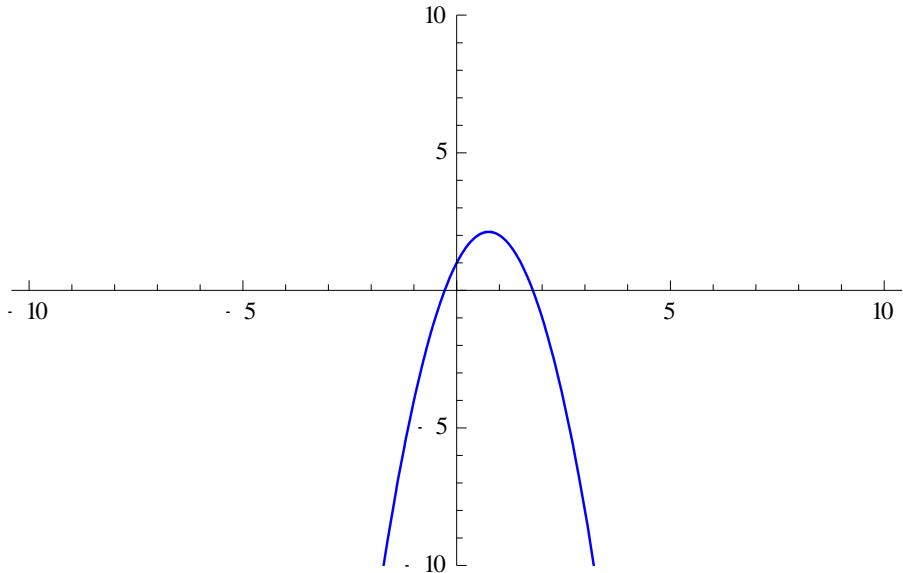


Figure 2

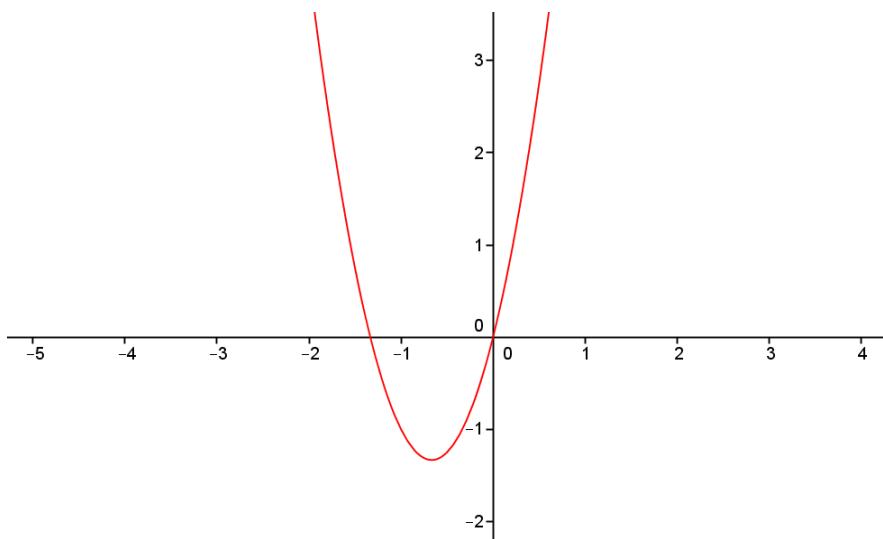


Figure 3

*Example 2:*

Examine and graph the next functions:  $f_1(x) = \frac{x-2}{x-3}$ ,  $f_2(x) = \frac{x+5}{x-4}$ ,  $f_3(x) = \frac{x}{x+1.5}$ ,  $f_4(x) = \frac{x-2.5}{x}$ ,  
 $f_5(x) = \frac{x-0.5}{x+3.5}$ .

*Solution:*

All of these five functions are the special cases of the function  $f(x) = \frac{x-a}{x-b}$ , where  $a$ ,  $b$  and  $c$  are real parameters. Function  $f(x)$ , as a parent function for all the functions above, can be examined in Mathematica or GeoGebra environment.

Click on [WM2](#) opens the window into the software package Wolfram Mathematica with interactive display of the function  $f(x) = \frac{x-a}{x-b}$ .

Click on [GGB2](#) opens the window into the software package GeoGebra with interactive display of the function  $f(x) = \frac{x-a}{x-b}$ .

*Example 3:*

Examine and graph the next functions:  $f_1(x) = \frac{(x-2)^3}{(x+2)^3}$ ,  $f_2(x) = \frac{(x+1)^2}{(x-1)^2}$ ,  $f_3(x) = \frac{x-1.5}{x+1.5}$ .

*Solution:*

It is easy to notice that these functions belong to the same class. There are several possibilities for their parent function. One of them is  $f(x) = \frac{(x+a)^b}{(x-a)^b}$  where  $a$  and  $b$  are real parameters.

Click on [WM3](#) opens the window into the software package Wolfram Mathematica with interactive display of the function  $(x) = \frac{(x+a)^b}{(x-a)^b}$ .

Click on [GGB3](#) opens the window into the software package GeoGebra with interactive display of the function  $(x) = \frac{(x+a)^b}{(x-a)^b}$ .

**Example 4:**

Examine and graph the next functions:  $f_1(x) = \frac{e^{3x}}{x}$ ,  $f_2(x) = \frac{1}{x}$ ,  $f_3(x) = \frac{1}{x e^{2x}}$ .

**Solution:**

The simplest parent function for these three functions would be  $(x) = \frac{e^{ax}}{x}$ , where  $a$  is real parameter.

Click on [WM4](#) opens the window into the software package Wolfram Mathematica with interactive display of the function  $(x) = \frac{e^{ax}}{x}$ .

Click on [GGB4](#) opens the window into the software package GeoGebra with interactive display of the function  $(x) = \frac{e^{ax}}{x}$ .

## The tasks to test achievement

It is expected that described method of the function examining helps students to better understand the influence of parameters on the function's characteristics. But it is also necessary to verify how they understood it. For solving the following several tasks it is necessary to apply knowledge about the influence of parameters on the function's characteristics.

1. Determine the values of parameters  $a$  and  $b$  that each of the next graph corresponds to the function  $y = ax^b$ .

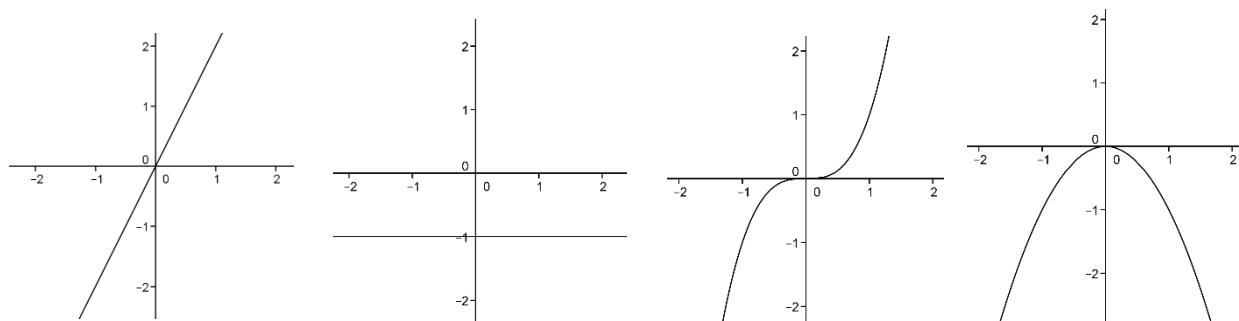


Figure 4

2. Observe the graphs in the following table. In which way are, in your opinion, these functions sorted in rows and columns of the table?

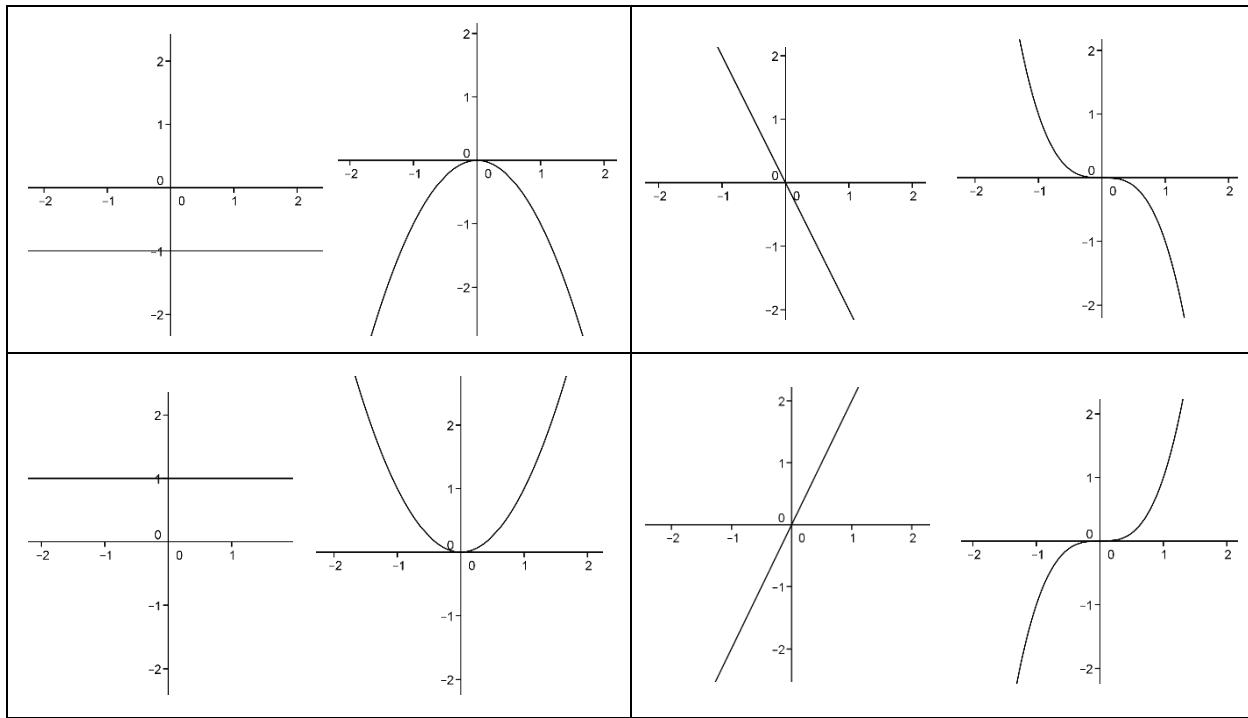


Table 1

3. Which of the following graphs could correspond to the function,  $y = \frac{a}{x}$ , where  $a$  is a real parameter?

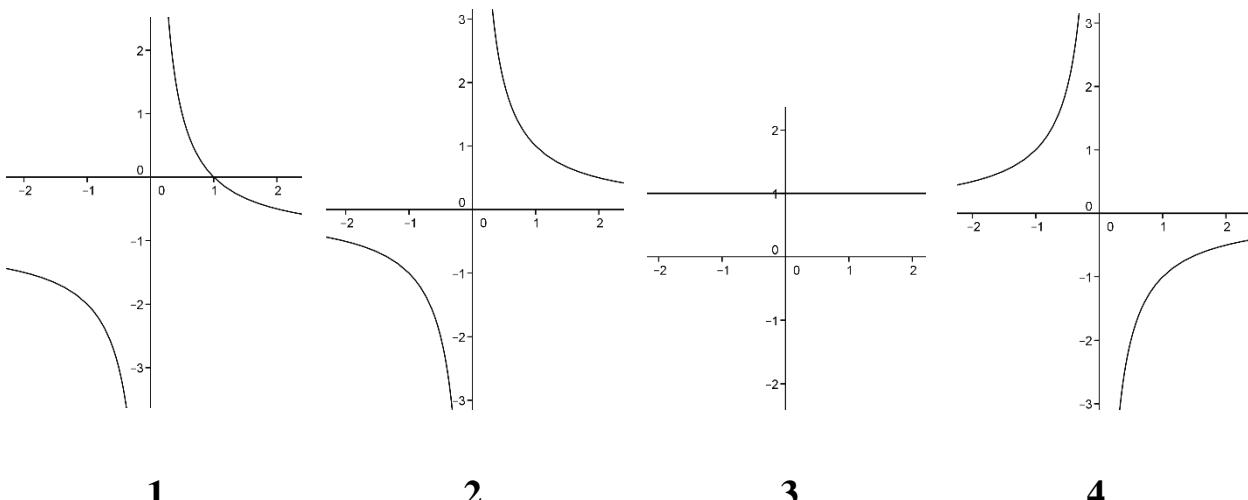


Figure 5

4. Determine the parent function for the following functions (if it is possible find more than one solution):  $f(x) = \frac{1}{(1+x)e^x}$ ,  $g(x) = \frac{1}{x}$ ,  $h(x) = \frac{e^{3x}}{x-3}$ .

5. Connect the corresponding functions on the following figure:

$$\frac{1}{x}$$

$$\frac{x^2-a}{(x-b)^2}$$

$$\frac{e^x}{x^2}$$

$$\frac{ax}{x+1}$$

$$(x+3)x^2$$

$$(x-a)(x-b)^2$$

$$\frac{x^2}{(x-3)^2}$$

$$\frac{e^{ax}}{x}$$

$$f(x) = 0$$

$$\frac{e^x}{a+x^2}$$

Figure 6

6. From the following list allocate the functions which could originate from the function

$$y = \frac{e^{ax}}{x-b} : \quad \frac{1}{xe^x}, \quad \frac{e^{3x}}{x^2-8}, \quad \frac{e^x}{x-5}, \quad \frac{1}{x}, \quad \frac{-9}{xe^x}, \quad \frac{1}{x+7}, \quad \frac{e^{2x}}{3x}$$

7. Which of the following graphs could correspond to the function  $y = e^{ax}$ ? Determine the value of the parameter  $a$  in that case?

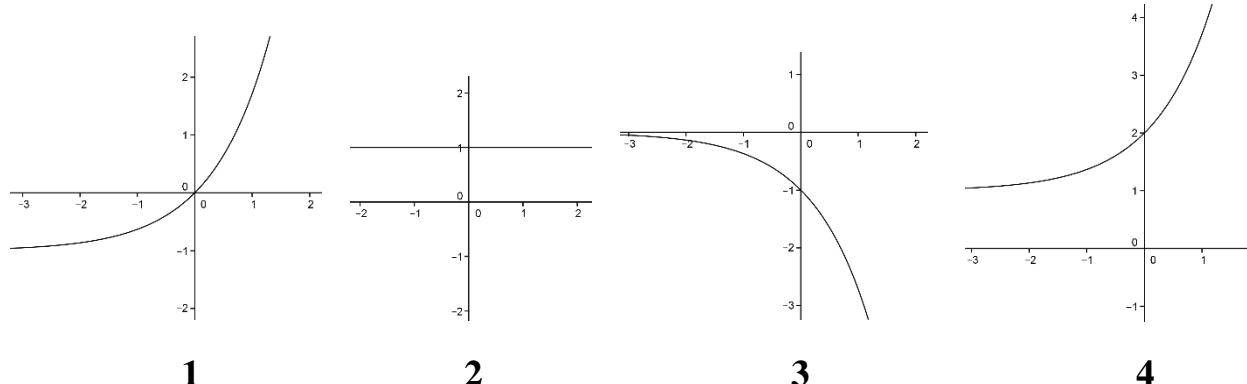


Figure 7

8. Which of the following graphs could corresponds to the function  $y = x^2 - a$ , and which one to the function  $y = (x - b)^2$ , where  $a$  and  $b$  are real parameters. Determine the values of the corresponding parameters for each graph.

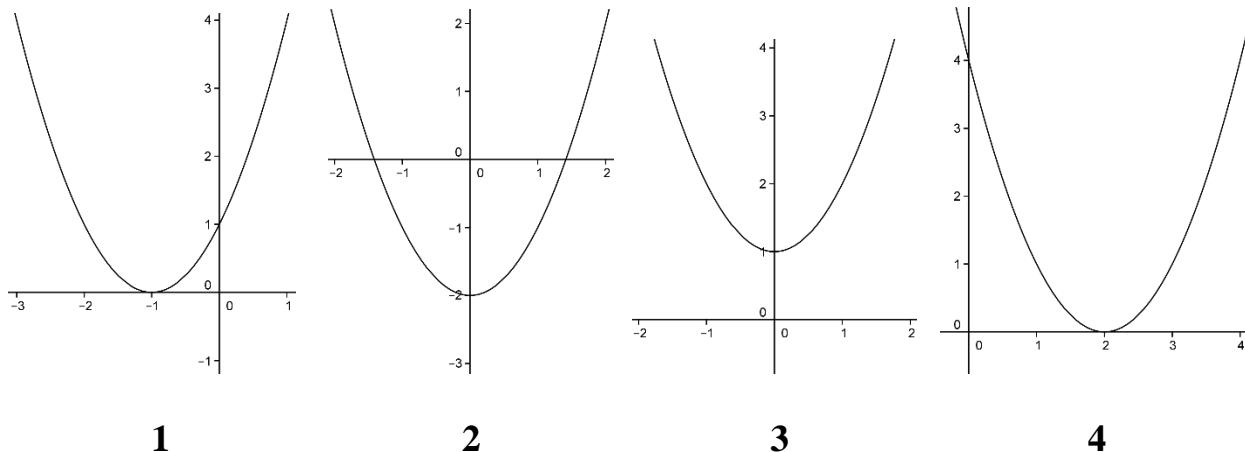


Figure 8

## Conclusion

Understanding of function characteristics is very important, not only for solving problems such as function examining and graphing, but also because of application of these contents in solving other problems, mathematical and also many problems from a real life, such a modeling by using mathematical functions, etc. Parameters which exist in each function have significant influence to its characteristics. It is very important for students to understand that influence and that is possible only by examining a lot of functions of the same classes and analyzing the influence of each parameter separately. It is very difficult and requires a lot of time to realize this without using any software. The use of dynamic software allows realizing of these analyze in a very short time and simultaneously comparison of several functions which belong to the same class. It is expected that the use of dynamic software will improve teaching process as far as function examining and graphing and help students to better understand characteristics of the concrete functions and an influence of parameters to these characteristics.

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